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Capital Growth Model**

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**DOCUMENTO DE TRABAJO 96-16**

May, 1996

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# Replacement Echoes in the Vintage Capital Growth Model\*

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May 1996

## Abstract

This paper is concerned with a non-standard source of fluctuations, called *echo effects*, i.e., the ability of an economy to reproduce its own past history. In the sixties, growth theorists believed that this property could arise in vintage capital models, taking the form of *replacement echoes*. This line of research was stopped after the publication of Solow *et al.* (1966), who showed that echoes should vanish in a Solow growth model with vintage capital. However, the fundamental reason for this result has nothing to do with vintages; rather, it comes directly from the main assumption in a Solow growth model, i.e., at equilibrium investment is a constant fraction of output and does not depend on any optimal replacement rule. We claim that in a more general Ramsey model replacement echoes do not necessarily vanish and we show this in the case when instantaneous utility is linear.

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\*The authors are indebted to A. d'Autume, J. Caballé, R. Caminal, R. Cooper, F. Portier, M. Santos, H. Uhlig and one anonymous referee for their stimulating comments. Of course, we are the solely responsible for any remaining errors. Financial support of the European Commission (Grant ERBCHRXCT940658) is gratefully acknowledged.

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# 1 Introduction

Modern economic theory gives essentially two different explanations for economic fluctuations: the existence of exogenous uncertainty or the existence of non-linearities. However, in the sixties growth theorists were concerned with a third source of fluctuations, so-called *echo effects*, i.e., the ability of an economy to reproduce its own past behavior. This property could arise in a vintage capital economy, where technological progress is embodied in new machines. At each period, firms must decide how many old machines must be scrapped and how much to invest in new machines. Replacement activity, i.e., the substitution of an old machine for a new one, is the key element of vintage capital models, and is the main cause of the existence of *replacement echoes*. When investment is mainly guided by replacement activity, investment today is high (resp. low) when we replace a high (resp. low) stock of old machines (i.e., past investment).

Although a great number of papers were written on vintage capital growth models in the 1960s,<sup>1</sup> a very few of them were really concerned with the dynamic properties of these economies. Moreover, Solow *et al.* (1966) and Sheshinski (1967), two of these very few exceptions, showed that replacement echoes should vanish in a Solow growth model with vintage capital (hereafter SVCM). In fact, this negative result is probably a mayor cause of the decline of the vintage capital literature in the 1970's and 1980's. However, as we will show in this paper, the fundamental reason for this result has nothing to do with vintages and it comes directly from the main assumption in a Solow growth model, i.e., at equilibrium investment is a constant fraction of output and does not depend on any optimal replacement rule.

It should be noted that the decline of vintage capital literature mentioned above can also be explained by the technical problems that confronted, and still confront, those who wish to go beyond the SVCM (in particular the constancy of the investment rate). Indeed, some further and more recent important developments along this research line have also made some crucial technical contributions, as for example the introduction of the differential-difference mathematical literature (Bellmand and Cooke (1963)) in economics. Using this mathematical approach, Benhabib and Rustichini (1991) established the existence of periodic solutions in the setup of optimal growth with vintage capital. However, these authors established this result for some exogenous physical or technical depreciation rules, e.g. the so-called "one-hoss shay

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<sup>1</sup>See for example Johansen (1959) and Solow (1962).

model.” In fact, the production technology considered in Rustichini and Benhabib, being more general than the Leontieff structure assumed in the SVCM, does not allow one to address the issue of endogenous replacement decisions straightforwardly.

In a discrete time optimal growth model with vintage capital, Benhabib and Rustichini (1993) consider the possibility of endogenous determination of scrapping rules under gross complementarity in technology. However, probably due to technical reasons, the authors do not characterize explicitly the replacement dynamics under endogenous scrapping. In order to be able to analyze the problem of endogenous scrapping in an optimal growth model, we come back to the Leontieff technology assumed in the sixties and we provide a benchmark for the study of endogenous replacement dynamics, which are not really studied in the cited Benhabib and Rustichini contributions.

According to the objectives mentioned just above, we study a continuous time Ramsey model with vintage capital and linear utility. Linear preferences are indispensable to bring out some explicit characterization of the solution paths. The model gives rise to corner and interior solutions. We fully characterize each solution and show that the economy converges to the interior solution at a finite distance, independently of its investment initial profile. At the interior solution, we show that detrended investment and output are purely periodic, that is to say echoes do occur in the long run in the linear Ramsey vintage capital growth model (hereafter LVC). We conclude that the presence of vintages may well cause replacement echoes in optimal growth models, in which the savings rate is not assumed constant.

The paper is organized as follows: section 2 describes and briefly analyzes the SVCM, while section 3 provides the formulation of the LVC. Section 4 solves for the interior solution of the latter model and section 5 describes its corner solutions and the transition dynamics to the interior solution starting with any initial investment profile. Section 6 shows how replacement echoes should occur starting at a finite date and section 7 concludes.

## 2 The Solow Vintage Capital Model

The SVCM is described and analyzed in Solow *et al.* (1966) and Sheshinski (1967). It is defined by the following block recursive equation system,  $\forall t \geq 0$ :

$$y(t) = \int_{t-T(t)}^t i(z) dz \quad (1)$$

$$\int_{t-T(t)}^t i(z) \exp\{-\gamma z\} dz = 1 \quad (2)$$

$$i(t) = s y(t) \quad (3)$$

$$w(t) = \exp\{\gamma(t - T(t))\} \quad (4)$$

$$1 = \int_t^{t+J(t)} (1 - w(z) \exp\{-\gamma t\}) R(z - t) dz \quad (5)$$

$$J(t) = T(t + J(t)) \quad (6)$$

with initial conditions  $i(t), \forall t < 0$ .

$y(t)$  is production,  $i(t)$  is investment,  $T(t)$  represents the age of the oldest operating machines or *scrapping time*,  $w(t)$  is the wage rate,  $R(t)$  is the discount factor and  $J(t)$  represents the *expected scrapping time* for machines bought at time  $t$ . In order to reproduce the Kaldor stylized growth facts, technical progress is supposed to be labor augmenting or *Harrod neutral* by assuming that each machine of generation  $t$  has a labor requirement of  $\exp\{-\gamma t\}$  workers. The parameter  $\gamma$  is supposed to be positive and represents technical progress.

Equations (1) to (3) represent the equilibrium in both the goods and the labor market. The vintage technology is in equations (1) and (2). Machines from different *vintages* are supposed to produce one unit of output each and to require  $\exp\{-\gamma t\}$  units of labor. The labor supply is assumed to be constant and equal to one for simplicity. The key assumption in the Solow growth model is that at equilibrium, a proportion  $s$  of output (the saving rate) is saved and invested.

Equation (4) states that the equilibrium real wage is equal to the marginal productivity of labor, i.e., the inverse of the labor requirement of the oldest (and marginal) operating machines. Finally, equation (5) corresponds to the optimal investment rule and states that the marginal cost of investing, which is equal to one, must be equal to the marginal revenue, which depends on the future scrapping of new machines. Equation (6) is just a definition, i.e., the expected life time for new machines  $J(t)$  is equal to the scrapping time  $T(\cdot)$  evaluated at  $t + J(t)$ , which correspond with the time at which these new machines will be scrapped in the future.

An important property of the SVCMM is its block recursive structure, i.e., the *backward* part of the system, equations (1) to (3), can be solved first in  $y(t)$ ,  $i(t)$  and  $T(t)$ . In particular, the age of the oldest operating machines  $T(t)$  is solved first of all in equation (2) as a function of past investment only. Under the condition that  $s > \gamma$ , and some more technical assumptions,

Solow *et al.* show that the economy converges to a unique balanced growth path.<sup>2</sup> Consequently, the SVCM has the same qualitative dynamics as the standard Ramsey growth model with homogeneous capital.

Notice that the block recursive structure of the SVCM depends crucially on the key assumption of the Solow growth model, i.e., investment is a constant fraction of output. In a Solow growth model, households determine quantities and firms determine prices in the capital market. As a consequence, equilibrium investment is determined by the savings behavior of households and the optimal rule for investment, equation (5), is at the bottom of a block recursive system, and determines the interest factor  $R(t)$ .

However, since the publication of Solow *et al.*'s paper an important question remains still open: has the Ramsey model with vintage capital the same qualitative dynamics than the SVCM and then the standard Ramsey growth model? Unfortunately, even if we can write down the optimal conditions for the Ramsey growth model with vintage capital, we cannot solve it in general since the equilibrium conditions for this economy give rise to a mixed-delay differential equation system with endogenous leads and lags. As stated by Boucekkine *et al.* (1996), until our days, this type of dynamic systems cannot in general be solved either mathematically or numerically.

### 3 The Linear Ramsey Vintage Model

As we cannot solve the general Ramsey vintage capital growth model, we focus on the linear utility case and we try to bring out an analytical characterization of the solution paths. Let a central planner solve the following problem:

$$\max \int_0^{\infty} c(t) \exp\{-\rho t\} dt \quad (7)$$

subject to

$$y(t) = \int_{t-T(t)}^t i(z) dz \quad (1)$$

$$\int_{t-T(t)}^t i(z) \exp\{-\gamma z\} dz = 1 \quad (2)$$

$$c(t) = y(t) - i(t) \quad (8)$$

$$0 \leq i(t) \leq y(t) \quad (9)$$

given  $i(t)$  for all  $t < 0$ .

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<sup>2</sup>See Boucekkine *et al.* (1996) for a numerical analyses of the dynamic behavior of the SVCM.

In order to solve for this control problem we write down the Lagrangian function, denoted by  $\mathcal{L}_i$ . Following Malcomson (1975), after changing the order of integration, and some algebra, we get:

$$\begin{aligned} \mathcal{L}(t) = & \int_0^\infty y(t) (1 - \phi(t)) e^{-\rho t} dt + \int_0^\infty w(t) e^{-\rho t} dt - \\ & \int_0^\infty i(t) \left( 1 - \int_t^{t+J(t)} (\phi(z) - w(z) e^{-\gamma t}) e^{-\rho(z-t)} dz \right) e^{-\rho t} dt - \\ & \int_{-T(0)}^0 i(t) \left( 1 - \int_t^{t+J(t)} (\phi(z) - w(z) e^{-\gamma t}) e^{-\rho(z-t)} dz \right) e^{-\rho t} dt \end{aligned}$$

where

$$J(t) = T(t + J(t)). \quad (6)$$

$\phi(t)$  and  $w(t)$  are the Lagrangian multipliers associated with constraints (1) and (2) respectively. Our optimization problem is similar to the one discussed by Malcomson, except that our central planner structure may give rise to corner solutions.<sup>3</sup> To characterize corner and interior solutions we use a trivial tool, taking advantage of the linearity of the Lagrangian with respect to  $i(t)$  and  $y(t)$ . The optimal rules are,  $\forall t \geq 0$ :

$$\phi(t) = 1$$

$$w(t) = \exp \{ \gamma (t - T(t)) \} \quad (4)$$

$$i(t) \begin{cases} = 0 & \text{if } \Phi(t) < 1 \\ = y(t) & \text{if } \Phi(t) > 1 \\ \in [0, y(t)] & \text{if } \Phi(t) = 1 \end{cases} \quad (10)$$

where

$$\Phi(t) \equiv \int_t^{t+J(t)} (1 - e^{-\gamma(t-z+T(z))}) e^{-\rho(z-t)} dz.$$

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<sup>3</sup>A more detailed discussion of the necessary and sufficient optimality conditions can be found in Malcomson (1975). Actually, the unique problem relies on the concavity of the Lagrangian with respect to  $J(t)$ : a sufficient condition for a global optimum cannot be verified before deriving explicitly the solution for  $J(t)$  (as  $J(t)$  appears as an integration bound in the Lagrangian). Indeed, unlike in Malcomson, we are unable to check this condition analytically for a particular initial investment profile, as explained in details in subsection 5.2.

Conditions (10) describe the three possible regimes the economy can experience. At first, all the regimes are “feasible”: depending on the initial investment profile, any regime can occur at least at  $t = 0$ . To get an intuition of this, observe that the initial conditions,  $i(t)$  with  $t < 0$ , determine  $T(0)$  by (2), which is likely to matter in the inequalities, at  $t = 0$ , that characterize each of the three possible regimes. So, the basic mathematical problem of our model is that we cannot in general implement the interior solution beginning at  $t = 0$ , because the initial investment profile may not allow for it.

In the following sections, we focus on the dynamics of  $T(t)$ , since the block recursive structure of our problem allows us to solve first equations (2), (6) and (10) on  $T(t)$ ,  $J(t)$  and  $i(t)$ . Denoting the interior solution by  $(T^o(t), J^o(t), i^o(t))_t$ , we operate as follows:

- i) The Terborgh-Smith result. We show that, if the economy is sometime at the interior solution, then it stays on and both  $T^o(t)$  and  $J^o(t)$  are equal to the same constant function.
- ii) Starting with any initial investment profile, or alternatively with  $T(0)$  not necessarily equal to  $T^o$ , we show that the economy converges at a finite distance to the interior solution.
- iii) We characterize the complete dynamics of investment and production, and in particular we show how replacement echoes affect these dynamics.

Before conducting the operations described just above, we need some assumptions on the initial investment profile, some of them merely for convenience:

**Assumption 1**

$$\begin{aligned} \exists t^* < 0 \text{ such that } & i(t) > 0, \forall t \geq t^* && \text{“No holes”} \\ \text{and } & \int_{t^*}^0 i(z) \exp\{-\gamma z\} dz \geq 1 && \text{“full-employment”} \end{aligned}$$

**Assumption 2** “Piecewise continuity”

$\exists (t_1 = t^* < t_2 < t_3 \dots < t_n = 0) \in R_-^n$  such that:

- i)  $i(t)$  is continuous on  $]t_j, t_{j+1}[$   $\forall j \in \{1, 2, \dots, n-1\}$
- ii)  $\forall j \in \{1, 2, \dots, n-1\}$   $\lim_{t \rightarrow t_j^-} i(t) < \infty$
- $\forall j \in \{1, 2, \dots, n-1\}$   $\lim_{t \rightarrow t_j^+} i(t) < \infty$



The “no holes” assumption will allow us to use some results established by Solow *et al.* (1966). The “full-employment” assumption is a sufficient condition to ensure that equation (2) makes sense at  $t = 0$ . Assumption 2 seems a little bit less general than in Solow *et al.*, who only assumes integrability. But in our framework, this hypothesis is necessary to get some explicit results.

The next section is devoted to the characterization of the interior solution, solving for function  $T(t)$  and  $J(t)$  in the set of continuous, piecewise differentiable functions, say in the  $CD_1^p$  set.  $T(t)$  and  $J(t)$  are assumed continuous, differentiable functions in open intervals. They can be non-differentiable at some points where the one-sided derivatives, while **finite**, can be different. Our assumption is indeed more general than the one adopted (i.e. global differentiability) in analogous frameworks by Malcomson (1975) and van Hilten (1991). Obviously, it fits better the structure of our model since the transition from a regime to another can involve pointwise non-differentiabilities.

## 4 Characterizing the interior solution

This section follows Boucekkine *et al.* (1995). Let us assume that there exists  $0 \leq t_0 < +\infty$  such that the interior solution is implementable beginning at  $t = t_0$ . From the optimal condition (10), we can focus first on the determination of  $T(t)$  and  $J(t)$  in this regime, which are determined by the system:

$$\Phi(t) \equiv \int_t^{t+J(t)} \left[ 1 - e^{-\gamma(t-z+T(z))} \right] e^{-\rho(z-t)} dz = 1 \quad (11)$$

$$J(t) = T(t + J(t)) \quad (6)$$

Provided that  $T(t)$  and  $J(t)$  are in the set  $CD_1^p$ , we can state the following proposition:

**Proposition 1** *For a given  $t > t_0$ ,  $J(\cdot)$  differentiable at  $t$ ,  $J(t)$  and  $T(t)$  are such that*

$$T(t) = F(J(t))$$

with

$$F(x) = -\frac{1}{\gamma} \ln \left[ 1 - \rho + \gamma - \frac{\gamma}{\rho} + \frac{\gamma}{\rho} \exp\{-\rho x\} \right]$$

provided that  $F(x)$  is defined  $\forall x \geq 0$ .

Given that  $J(t)$  is differentiable at  $t$ , we can differentiate (11) and easily show Proposition 1 after some elementary manipulations.

Now, in order to establish the Terborgh-Smith result, i.e.  $T^\circ(t) = J^\circ(t) = T^\circ$  with  $T^\circ$  a specific constant, we make a further restriction on the parameters values:

**Assumption 3** Parameters  $\gamma$  and  $\rho$  must hold the following condition:

$$0 < \gamma < \rho < 1.$$

Assumption 3 is a standard assumption for the existence of solutions in exogenous growth models. In fact, this assumption will allow us to use a fixed-point argument à la van Hilten (1991) in order to establish the Terborgh-Smith property. This argument requires function  $F(\cdot)$  being strictly increasing and admitting a unique strictly positive fixed-point. It is easy to show that under Assumption 3,  $F(\cdot)$  fulfills the latter requirements.

We are able now to show the Terborgh-Smith property. First, we prove it by assuming global differentiability of  $T(t)$  and  $J(t)$  as in van Hilten, then we provide the extension of the proof to the case of  $CD_1^p$  functions.

**Proposition 2** *Under Assumption 3, the unique differentiable interior solutions  $T(t)$  and  $J(t)$ ,  $t \geq t_0$ , are defined by*

$$T(t) = J(t) = T^\circ$$

with  $T^\circ$  the positive fixed-point of function  $F(\cdot)$ .

**Proof:** By Proposition 1, we know that  $T(t) = F(J(t))$ ,  $\forall t > t_0$ . By continuity of  $T(\cdot)$ ,  $J(\cdot)$ , and  $F(\cdot)$ , we deduce  $T(t) = F(J(t))$ ,  $\forall t \geq t_0$ . Since  $J(t) \geq 0$  and  $F(x) > 0$ , we have  $F(0) \leq T(t) \leq \bar{F}$ , for every  $t \geq t_0$ , with  $\bar{F} = \lim_{x \rightarrow +\infty} F(x)$ .

Applying the latter inequalities at  $t + J(t)$  and using (6) yields  $F(0) \leq J(t) \leq \bar{F}$ ,  $\forall t \geq t_0$ . As  $F'(x) > 0$ , for  $x \geq 0$ , we get  $F(F(0)) \leq T(t) \leq F(\bar{F})$ ,  $\forall t \geq t_0$ .

As before, we can apply the latter inequalities at  $t + J(t)$ , find new lower and upper bounds for  $J(t)$  and then for  $T(t)$ . Repeating this reasoning, we can construct a sequence of lower ( $X_n$ ) and upper bounds ( $Y_n$ ) for  $T(t)$ ,  $\forall n \geq 0$  and  $\forall t \geq t_0$ , such that:

$$X_n \leq T(t) \leq Y_n$$

where

$$\begin{aligned} X_0 = F(0) & \quad \text{and} \quad X_n = F(X_{n-1}), \quad n \geq 1 \\ Y_0 = \bar{F} & \quad \text{and} \quad Y_n = F(Y_{n-1}), \quad n \geq 1 \end{aligned}$$

It is easy to show that  $(X_n)$  is an increasing bounded sequence and that  $(Y_n)$  is a decreasing bounded sequence. Both of them converge by construction to the fixed-point of  $F(\cdot)$ ,  $T^\circ$ .  $\square$

The extension to the  $CD_1^p$  case uses the following lemma:

**Lemma 1** *If  $J(\cdot)$  is differentiable at  $t, t > t_0, J(\cdot)$  is differentiable at  $t+J(t)$ .*

**Proof:** If  $J(t)$  is differentiable at any  $t > t_0$ , then:  $T(t) = F(J(t))$ .  $F(\cdot)$  being a diffeomorphism, we have the equivalence:

$$J(t) \text{ differentiable at } t \iff T(t) \text{ differentiable at } t$$

On the other hand, differentiating (6) yields:

$$T'[t + J(t)] = \frac{J'(t)}{1 + J'(t)}$$

So if  $J'(t)$  exists,  $T'[t + J(t)]$  exists and so should exist  $J'[t + J(t)]$  given the equivalence before.

The case  $J'(t) = -1$  is of course excluded by the definition of  $CD_1^p$  functions as it involves an infinite derivative of  $T(\cdot)$  at  $t + J(t)$ .  $\square$

Then, we can state the Terborgh-Smith property in the  $CD_1^p$  case:

**Proposition 3** *Under Assumption 3, the unique  $CD_1^p$  solutions  $T(t)$  and  $J(t)$  are given by:*

$$T(t) = J(t) = T^\circ \quad \forall t \geq 0$$

*with  $T^\circ$  the positive fixed-point of  $F(\cdot)$ .*

**Proof:** The proof of Proposition 2 consists in constructing two sequences of lower and upper bounds exclusively using the fact that if  $T(t) = F(J(t))$  then  $T(t + J(t)) = F(J(t + J(t)))$  under the assumption that both  $T(t)$  and  $J(t)$  are globally differentiable for  $t \geq t_0$ . In the  $CD_1^p$  case, we get exactly the same property if we prove that the differentiability of  $J(\cdot)$  at  $t$  implies the differentiability of  $J(\cdot)$  at  $t + J(t)$ . The latter implication is established in the lemma just above. We can consequently conclude that:  $T(t) = J(t) = T^\circ$  for any point  $t$  such that  $J(\cdot)$  is differentiable at  $t$ . By definition of the  $CD_1^p$  set (especially the continuity requirement), we get:  $T(t) = J(t) = T^\circ, \forall t \geq t_0$ .  $\square$

It remains to see why the interior solution is reached at a finite distance, or in other words, why it exists  $0 \leq t_0 < +\infty$  such that the equality condition (11) holds at  $t_0$ . This is done in the following section.

## 5 The transition dynamics

Given the initial investment profile,  $T(0)$  solves equation (2) -the labor market equilibrium condition.  $T(0)$  exists and is unique by Assumption 1. However, three cases are possible depending on the position of  $T(0)$  with respect to  $T^o$ . The case  $T(0) = T^o$  is trivial since it allows us to implement the interior solution beginning at  $t = 0$ . The two other cases are discussed below:

### 5.1 A too high initial stock of machines: $T(0) < T^o$

In this case, at  $t = 0$  more machines are scrapped than there would be in the interior solution. Intuitively, this should correspond to the case where the economy starts with a relatively too high stock of machines. It corresponds to the  $i(0) = 0$  regime, described by the corresponding inequality in condition (10) at  $t = 0$ . This intuitive property is established in the following proposition:

**Proposition 4** *If  $T(t) < T^o$ , then*

$$\Phi(t) \equiv \int_t^{t+J(t)} \left[ 1 - e^{-\gamma(t-z+T(z))} \right] e^{-\rho(z-t)} dz < 1. \quad (12)$$

**Proof:** First, note that if  $T(t) < T^o$ , then  $J(t) \leq T^o$ . Indeed, by the forward-looking condition (6) we have:  $J(t) = T(t + J(t))$ . If  $J(t) > T^o$ , by continuity there exists  $t_0$ ,  $t < t_0 < t + J(t)$ , such that  $T(t_0) = T^o$ . Then, for every  $t \geq t_0$ ,  $T(t) = T^o$  is the optimal solution. Which contradicts the assumption:  $T(t + J(t)) = J(t) > T^o$ .

Consider the case  $J(t) = T^o$ . From Proposition 3,

$$\int_t^{t+T^o} \left[ 1 - e^{-\gamma(t-z+T^o)} \right] e^{-\rho(z-t)} dz = 1 \quad (13)$$

Then the inequality (12) holds if and only if:

$$\int_t^{t+J(t)} \left[ e^{-\gamma(T(z)-z)} - e^{-\gamma(T^o-z)} \right] e^{-(\rho+\gamma)z} dz > 0,$$

which is obvious given that  $T(0) < T^o$  and  $T(t) \leq T^o$  on the interval  $[0, T^o]$  (remember that  $T(t)$  cannot become greater than  $T^o$ ).

It remains to address the case  $J(t) < T^o$ . Decompose the left hand side of equality (13), into the sum of two integrals, the first with the integration bounds  $t$  and  $t + J(t)$  and the second with the integration bounds  $t + J(t)$  and  $t + T^o$ . The first integral is greater than the left hand side of inequality

(12) by the same argument as in the case  $J(t) = T^o$ . The second integral of the decomposition is strictly positive. Then, (12) also holds in this last case.  $\square$

We characterize now explicitly the transition dynamics when the regime given by inequality (12) prevails at the initial period and we show precisely how and when the interior solution is reached.

**Proposition 5** *For a given initial investment profile, under Assumptions 1 to 3, if  $T(0) < T^o$ , then:*

i) *The interior solution is reached at  $t_0 = T^o - T(0)$ ;*

ii)  $\forall t, 0 \leq t < t_0 \quad T(t) = t + T(0)$   
 $\forall t, t \geq t_0 \quad T(t) = T^o$ ;

iii)  $\forall t \geq 0 \quad J(t) = T^o$

**Proof:** Since we assume  $T(0) < T^o$ , by Proposition 4 we know that  $\Phi(0) < 1$ . By continuity, there exists  $t_0$  such that  $T(t) < T^o, \forall t \in [0, t_0[$ . From condition (10)  $i(t) = 0, \forall t \in [0, t_0[$ , implying

$$\int_{t-T(t)}^0 i(z)e^{-\gamma z} dz = 1,$$

$\forall t \in [0, t_0[$ .

Using the latter characterization of this regime, the piecewise continuity of the initial investment profile and specially the “no-holes” assumption, it is very easy to show that  $t - T(t) = -T(0), \forall t \in [0, t_0[$ . Hence, the interior solution is reached at a finite distance, precisely at  $t_0 = T^o - T(0)$ .

To end the proof, we must show that:  $J(t) = T^o, \forall t \in [0, t_0[$ .

First, we show that  $t + J(t) \geq t_0, \forall t \in [0, t_0[$ . Assume that there exists  $t'$  such that  $t' \in [0, t_0[$  and  $t' + J(t') < t_0$ . By (6), we know that  $J(t') = T(t' + J(t'))$ , and  $J(t') = t' + J(t') + T(0)$  by property ii) of this proposition. So  $t' + T(0) = 0$  which is impossible as  $T(0) > 0$ . Hence  $\forall t \in [0, t_0[, t + J(t) \geq t_0$ .

Applying again equation (6) and using property ii) of this proposition, we get:

$$J(t) = T(t + J(t)) = T^o, \forall t \in [0, t_0[. \square$$

## 5.2 A too low initial stock of machines: $T(0) > T^o$

As in the previous case and using exactly the same type of arguments, we can show that if  $T(0) > T^o$ , then the regime described by  $\Phi(t) > 1$  should hold in a neighborhood of  $t = 0$ . This regime describes the cases where the economy starts with a relatively low stock of machines, which makes optimal, at the beginning, both to use older machines than in the interior solution, and to invest all production in new machines. Unlike in the previous subsection, now it is not possible to characterize explicitly the transition dynamics. However, we will show that the economy, starting with this regime, should converge at a finite distance to the interior solution. To this end, we use a powerful preliminary result proved by Solow *et al.* (1966) in their seminal paper. We state it as follows:

**Proposition 6** *Under Assumption 1 - "no holes" assumption- and if the following equations hold for  $t \geq 0$ :*

$$i(t) = y(t) = \int_{t-T(t)}^t i(z) dz \quad (14)$$

and

$$\int_{t-T(t)}^t i(z) e^{-\gamma z} dz = 1 \quad (2)$$

Then:

- i)  $T(t)$ ,  $i(t)$  and  $y(t)$  are differentiable  $\forall t \geq 0$
- ii)  $\lim_{t \rightarrow +\infty} T(t) = T^*$ , where  $T^* = \frac{-1}{\gamma} \ln(1 - \gamma)$ .

Observe that when the regime described by  $\Phi(t) > 1$  holds, we get exactly the system (14) - (2) stated in the proposition. This system is a special case of the model studied by Solow *et al.*: The exogenous saving rate is in our case equal to 1 and the growth rate of the population is zeroed.

Property i) is demonstrated by Solow *et al.*, remark 3 page 91. Property ii) is proved in their section 5, pp 94-98.

The finite distance convergence to the interior solution of our optimal control problem follows from the fact that  $T^* < T^o$ , as one can check easily. Since  $T(0) > T^o$ , the property  $T^* < T^o$  implies that there exists a finite  $t_0$  such that  $T(t_0) = T^o$ .

We can now give a general characterization of  $T(t)$  dynamics:

**Proposition 7** *For a given initial profile of investment, under Assumptions 1 to 3, if  $T(0) > T^o$ , then:*

*i) The interior solution is reached at finite  $t_0$  such that:*

*$t_0 = \inf \{t' > 0 \mid T(t') = T^o\}$  where  $T(t)$  solves the system (14)-(2).*

*ii)  $\forall t, 0 \leq t < t_0, T(t)$  solves the system (14)-(2).  $\forall t \geq t_0, T(t) = T^o$ .*

**Proof:** The proof is trivial given the arguments above.

Unfortunately, we are unable to give an equally accurate characterization for  $J(t)$ , except that  $J(t) = T^o, \forall t \geq 0$  if  $t_0 \leq T^o$ , or  $\forall t \geq t_0 - T^o$  otherwise. The equation  $J(t) = T(t + J(t)) \forall t \geq 0$  determine  $J(t)$  elsewhere given the solution  $T(t), t \geq 0$  stated in the proposition above. This problem comes from the unavailability of an explicit  $T(t)$  solution on  $[0; t_0[$ . As shown by Boucekkine *et al.* (1996), equations (14) and (2) -a particular case of the SVCM, yields a system of nonlinear non-autonomous differential-difference equations that can be solved only numerically using some very advanced computational mathematics tools. Hence, it is not possible to give an explicit characterization for  $T(t)$  in the transition regime, which in turn disables us to be more precise on  $J(t)$ 's dynamics in this transition regime.

We end now our dynamic analysis by studying the investment and production solutions.

## 6 Investment and production dynamics: replacement echoes

In this final section, we explain how replacement echoes can arise in this framework. More precisely, we prove that optimal production and investment paths show periodicity beginning at a well-defined date. When the interior solution is reached and for any initial investment profile, given the characterization of  $T(t)$  dynamics studied above, investment dynamics follows directly from equation (2). Production dynamics come mainly from equation (1) once investment dynamics are found out. As one can easily check, a difficulty comes from the fact that,  $y(t)$ 's solution depends on the position of  $t_0$  with respect to  $T^o$ ,  $t_0$  being the date at which the interior solution is reached using the notations of the previous section. When the economy starts with  $T(0) < T^o$ , we do not face this problem since we know the explicit value of  $t_0$ ,  $t_0 = T^o - T(0) < T^o$ . But when the economy starts with  $T(0) > T^o$ , we cannot characterize explicitly the date  $t_0$  for the precise reasons exposed in subsection 5.2 (namely, the impossibility of analytical resolution of nonlinear non-autonomous differential-difference equations that

yields the model in this regime). This point does not suppose any theoretical problem since we can derive the dynamics of production conditionally to the position of  $t_0$  with respect to  $T^o$  and find out the periodicity property for each case.

Investment dynamics are given by the following proposition:

**Proposition 8** *For any investment initial profile, under Assumptions 1-3 and denoting by  $t_0$  the date at which the interior regime is reached*

*i) For  $t \geq t_0$ ,  $i(t) = i(t - T^o)e^{\gamma T^o}$  provided  $i(t)$  is continuous at  $t - T^o$ .*

*ii) If  $t_0 > 0$ , for  $0 \leq t < t_0$ :*

*If  $T(0) > T^o$ ,  $i(t)$  solves the system (14)-(2),  $i(t)$  is differentiable.*

*If  $T(0) < T^o$ ,  $i(t) = 0$ .*

The investment solutions given by Property ii) are true by construction of the two non-interior regimes. The differentiability of  $i(t)$  during the transition from the regime characterized by  $T(0) > T^o$  to the interior solution is one consequence of Solow *et al.* findings (See Proposition 6 above). Property i) can be obtained simply by differentiation of equation (2) provided that  $i(t)$  is continuous at  $t - T^o$ , using the fact that  $T(t) = T^o$  after  $t_0$  by construction of  $t_0$ . Observe that property i) is true except at a finite number of points, given the piecewise continuity of the initial investment profile and given Property ii) of the proposition.

We give the production dynamics in the following proposition:

**Proposition 9** *For any investment initial profile, under Assumptions 1-3 and denoting by  $t_0$  the date at which the interior regime is reached*

*For  $t \geq t_0 + T^o$ ,  $y(t) = y(t - T^o)e^{\gamma T^o}$ .*

The proof of the proposition is trivial. As argued before, the main property of production dynamics, namely the periodic behavior, is obtained after  $t_0 + T^o$  independently of the position of  $t_0$  with respect to  $T^o$ . This property is obtained by differentiation of the integral equation (1) using the fact that over the integration range  $i(t)$  is periodic by Proposition 8. As one can see, the existence of echoes in production is a consequence of the existence of investment replacement echoes.

## 7 Conclusions

In this paper we have fully characterized the dynamics of the vintage capital Ramsey model with instantaneous linear utility and we have also shown that given any initial investment profile, this model gives rise to a periodic pattern



starting at a well-defined date. It is very important to notice that, even under the assumption of linearity on preferences and technology, the resolution of the model is far from straightforward. Concerning the permanent regime (the interior solution), it uses a non-standard rational expectations argument to prove the constancy of the scrapping age (as in van Hilten (1991)) and takes advantage of this property to obtain periodicity (as in Benhabib and Rustichini (1991)). Concerning the analysis of the transitional dynamics: first, we use some convergence results proved by Solow *et al.* (1966), using very sophisticated mathematical arguments; and finally, we show that some marginal properties (like the exact date at which the interior solution is reached when the initial capital stock is “too” low) are not analytically tractable.

Since the early 1990s there is a growing literature on vintage capital economies, with some authors analyzing structural problems as in Chari and Hopenhayn (1991) and Aghion and Howitt (1994) and some others analyzing cyclical problems as in Caballero and Hammour (1994), Cooley *et al.* (1994) and Gilchrist and Williams (1996). The LVC model solved in this paper could be seen as a benchmark for the analysis of growth and cycles in continuous time growth models with vintage capital. It seems clear that some very important extensions of the model, such that the analysis of the nonlinear utility case, cannot be undertaken without a prior resolution of the induced mathematical and computational problem. As reported by Boucekine *et al.* (1996), it is likely that these extensions will not allow in general even for a partial analytical treatment, but instead will require full numerical approaches. Unfortunately, the dynamic systems involved are so complicated that even numerical appraisal requires very sophisticated tools, still to be conceived and tested.

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