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Solution to Non-Linear MHDS arising from Optimal Growth Problems

by

J. R. Ruiz-Tamarit*
M. Ventura-Marco*

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* University of Valencia.

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Abstract

In this paper we propose a method for solving in closed form a general class of non-linear modified Hamiltonian dynamic systems (MHDS). This method may be used to analyze some intertemporal optimization problems with a predetermined structure involving unbounded technological constraints. The method seems specially well designed to study endogenous growth models with two controls and one state variable. We use the closed form solutions to study either unicity or indeterminacy of the non-explosive paths in a context characterized by the lack of a well defined isolated steady state. Moreover, in this way we can avoid both the reduction of dimension and the linearization process even when the dynamic system offers a continuum of steady states or no steady state at all.

JEL classification: C61, C62, O41.

Keywords: Non-Linearity, Hamiltonian Dynamic System, Closed Form Solution, Endogenous Growth, Transitional Dynamics.

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†Department of Economic Analysis. University of València. Spain.

‡Department of Financial and Mathematical Economics. University of València. Spain.

1 Introduction

Because of the lack of a general solution method for non-linear dynamic systems, techniques based on qualitative analysis have been developed and widely applied to study dynamic economic models. However, quantitative techniques have been also used to analyze both the linear systems which may be solved explicitly in a closed form, and the non-linear systems that we can try to solve applying different numerical solution methods[1]. As a general procedure we first solve an intertemporal optimization problem, which produces a non-linear dynamic system. Then, a linearization method is applied to transform that system. Finally, a combination of qualitative and quantitative methods are used to solve the resulting linear system[2]. By solving the system we mean to cover different stages like characterizing the steady state, studying transitional dynamics and eventually doing comparative dynamics. In this context, only very general requirements concerning the functions involved in the optimization problem are needed to ensure satisfactory mathematical properties. The bulk of the unpleasant arithmetic goes with the linearization process[3].

The previous procedure is particularly suitable for the study of many economic problems. However, it works appropriately only when the dynamic system may be characterized by the existence of a unique or multiple, but isolated, structurally stable steady state(s). Dynamic systems coming from optimal control problems[4] do not guarantee necessarily the above property. In fact, these dynamic systems always have as many initial conditions as predetermined variables as well as many transversality (terminal) conditions as non-predetermined costate variables, but only under certain particular circumstances they adopt a saddle point structure at the relevant steady state. In that case an opportune dynamical stability/instability dichotomy is en-

[1] See Medio (1992), Boucekkine (1995) and Judd (1998), for a general approximation to these two techniques. The particular methods of time elimination and backward integration which apply to deterministic perfect foresight dynamic models, may be seen respectively in Mulligan and Sala-i-Martín (1993) and Brunner and Strulik (1998).

[2] See Brock and Malliaris (1989) and Farmer (1993), as well as the related bibliography refered to there.

[3] For an alternative procedure, the so called Linear-Quadratic approximation, which proceeds by "quadrifying" the objective functional in the optimization problem in order to have the optimal decision rules directly in linear form, see Hansen and Prescott (1995).

[4] See specially Kurz (1968) who studies the dynamic properties of modified hamiltonian dynamic systems formulated in continuous time.

sured, such that the optimal stable path is unique and the economy converges to the steady state whenever the non-predetermined variables initially jump to the stable manifold.

Apart from the saddle point structure, other structures like nodes, spirals and centers, unstable or stable, asymptotic or not, have also been studied in economics. These alternative structures are able to report the presence of growth paths without bound, indeterminacy, or even cycles which may be sustained or not. All of them share a common feature: the steady state is always structurally stable. However, given the non-linearity of the original dynamic system, in the case where the linearization is undertaken, the Hartman-Grobman theorem establishes a serious limitation that deserves a deeper scrutiny. It is well known that, on the one hand, when the determinant of the Jacobian matrix is zero the steady state could not exist or there could be a continuum of such states. On the other hand, when the Hartman-Grobman theorem does not hold, which indicates that the non-linear dynamic system is structurally unstable in its steady state, the linear approximation cannot be used to study the dynamic behavior of the system either globally or locally.

In this paper we propose a method for solving in closed form a general class of non-linear modified Hamiltonian dynamic systems (MHDS). This method may be used to analyze some intertemporal optimization problems with the only proviso that they must conform to a certain predetermined structure. In particular, we have found examples of endogenous growth models with two controls and one state variable where the method can appropriately be applied[5]. In such cases, where the technological constraint exhibits constant returns to the single accumulable factor, the objective function has to combine the CRRA and the Cobb-Douglas functional form to make compatible both feasibility and optimality of balanced growth paths[6].

An interesting advantage of our procedure is that we can avoid the step

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[5]Within one sector endogenous growth models we lack stability and, in some cases, also stationarity. In the case of the standard AK optimal growth model, the economic system may be represented by an almost linear dynamic system of order two, written in the state and costate variables. However, it becomes linear when, using the first order condition, we rewrite it in the control and state variables. On the other hand, this possibility does not exist when the model includes more controls than costate variables.

[6]The functional form of the utility function when the variables are growing even at the (hypothesized) steady state, has been studied in King, Plosser and Rebelo (1987) and Bovenberg and Smulders (1995).

corresponding to the linear approximation in analyzing non-linear dynamic systems. Consequently, we do not need to do any boring algebraic manipulation, or accept the limitations imposed by this kind of approximation. Moreover, we can learn much about structurally unstable dynamic systems characterized by either the existence of a continuum of steady states or the absence of any steady state. In this context, the remaining sections are organized as follows. Section 2 introduces the modified Hamiltonian dynamic system and establishes a connection with the family of intertemporal optimization problems from which it could be derived. In section 3 we describe the solution method for this dynamic system, and prove for a particular set of parameter values that the non-explosive trajectories are unique in a context of general asymptotic instability. This section is complemented with an appendix where the issue of indeterminacy is discussed under an alternative set of parameter values for which the dynamic system is globally asymptotically stable. Then, in section 4, we study whether a steady state exists or not, characterize parametrically the different feasible situations, and analyze the optimal trajectories belonging to each one of such situations. Finally, in section 5 we provide some conclusions and outline directions for future research.

2 The MHDS

The dynamic system we are going to analyze takes the following form:

$$\dot{K}(t) = A \cdot K(t) - \varepsilon_k K(t)^{a_{11}} \lambda(t)^{a_{22}} \tag{1}$$

$$\dot{\lambda}(t) = D \cdot \lambda(t) + \varepsilon_\lambda K(t)^{a_{11}-1} \lambda(t)^{1+a_{22}} \tag{2}$$

with the two boundary conditions:

$$K(t_0) = K_0 \tag{3}$$

$$\lim_{t\to\infty} \lambda(t) K(t) \cdot \exp\{-\theta(t-t_0)\} = 0 \tag{4}$$

The elements $A > 0$, $D \geqslant -A$, $\varepsilon_k > 0$, $\varepsilon_\lambda \geqslant 0$, $a_{11} \gtreqless 0$, $a_{22} < 0$, K_0, t_0 and θ are constant parameters, while K, λ and t are the variables. Moreover, it is known that $\varepsilon_k > \varepsilon_\lambda$, $1 - a_{11} > 0$ and $1 + a_{22} \gtreqless 0$. This system may be classified as a modified Hamiltonian dynamic system (MHDS) as long as $A + D = \theta$, where θ represents a non-negative intertemporal rate of discount.

Although not strictly necessary from an analytical point of view, we can additionally assume that parameters are subject to the following constraints:

$$\frac{\varepsilon_k}{\varepsilon_\lambda} = \frac{1 + a_{22}}{a_{11}} \tag{5}$$

$$A > \theta > 0 \tag{6}$$

Under these more restrictive parameter constraints, all together, the above expressions define a modified Hamiltonian dynamic system arising from an optimal intertemporal resources allocation problem. In particular, consider the following formulation of what could be interpreted as an endogenous growth model with two controls and one state. First, given that this kind of models are focused on the problem of steady growth, we assume a time separable objective function with constant intertemporal elasticity of substitution and unitary elasticity of substitution between its contemporary determinants. Second, technology exhibits constant returns to the state variable. So, given the general interest in optimal outcomes, we will consider the planner's problem only. This consists in choosing the controls $u_1(t)$ and $u_2(t)$, $\forall t \geq t_0$ which solve the following optimization problem:

$$Max \int_{t_0}^{\infty} \frac{\left[u_1(t) \cdot (1 - u_2(t))^{\alpha}\right]^{1-\Phi}}{1 - \Phi} \cdot e^{-\theta \cdot (t - t_0)} \cdot dt \tag{P}$$

subject to $\overset{\bullet}{K}(t) = A \cdot K(t) \cdot u_2(t) - u_1(t)$, given $K(t_0) = K_0$. In this particular problem, the constant coefficient $\alpha > 0$ represents the weight of $1 - u_2(t)$ in utility relative to the weight of $u_1(t)$. The constant intertemporal elasticity of substitution is allowed to be either $\frac{1}{\Phi} \lesseqqgtr 1$.

Looking for applying the Maximum Principle we can build a Hamiltonian function introducing the costate variable $\lambda(t)$. Then, the optimality conditions for interior solutions give two control functions $u_1(t) = u_1(K(t), \lambda(t), \Omega)$ and $u_2(t) = u_2(K(t), \lambda(t), \Omega)$, which adopt a Cobb-Douglas form. In these functions Ω represents a vector of parameters. Substituting these functions in both the dynamic constraint and the Euler equation, $\overset{\bullet}{\lambda}(t) = \theta \cdot \lambda(t) - A \cdot u_2(t) \cdot \lambda(t)$, we get a MHDS similar to the system of equations (1) and (2). The initial boundary condition is complemented with a transversality condition, as in equations (3) and (4). As a particular case, this dynamic system corresponds to the system arising from the canonical AK optimal growth model, as studied in Rebelo (1991), when $\varepsilon_k = 1$, $\varepsilon_\lambda = 0$, $a_{11} = 0$, $a_{22} = -\frac{1}{\Phi}$.

Given the non-linearity of the dynamic system just considered, the interest and advantages of the method that we will develop in the following sections, become obvious from the moment we analyze the linear approximation around its steady state. In Appendix B we supply some calculations which allow us to conclude that the original system is structurally unstable and, consequently, that the linear approximation cannot be used to study the dynamic behavior of the non-linear system. On the other hand, the original dynamic system has an almost-linear structure. However, this fact is not as useful as could be expected. In Appendix C we study the dynamics of the linear part alone and, as will be shown, those results differ markedly from what may be deduced directly from the non-linear system as a whole.

3 The Solution

Now, we are ready to supply a method for solving in closed form dynamic systems like that of the previous section. The method works in three steps. First, we define the instrumental variable $X(t)$ as:

$$X(t) = K(t)^{1-a_{11}} \lambda(t)^{-a_{22}} \tag{7}$$

By totally differentiating the above equation we get:

$$\overset{\bullet}{X}(t) = (1 - a_{11})\, K(t)^{-a_{11}} \lambda(t)^{-a_{22}} \overset{\bullet}{K}(t) - a_{22} K(t)^{1-a_{11}} \lambda(t)^{-a_{22}-1} \overset{\bullet}{\lambda}(t)$$

which, after substituting equations (1) and (2), gives us the following non-homogeneous linear differential equation with constant coefficients:

$$\overset{\bullet}{X}(t) = a_x \cdot X(t) - b_x \tag{8}$$

The new parameters just introduced, $a_x \gtrless 0$ and $b_x \gtrless 0$, are related to the parameters of the dynamic system in the following way: $a_x \equiv A \cdot (1 - a_{11}) - D \cdot a_{22}$ and $b_x \equiv \varepsilon_k \cdot (1 - a_{11}) + \varepsilon_\lambda \cdot a_{22}$.

Given the initial condition (3) and a certain initial value $\lambda(t_0)$, although for the moment unknown, we can generate artificially an initial condition for $X(t)$. That is, $X(t_0) = K(t_0)^{1-a_{11}} \lambda(t_0)^{-a_{22}}$. Then, a particular solution to (8) will be of the form:

$$X(t) = \frac{b_x}{a_x} + \left[X(t_0) - \frac{b_x}{a_x}\right] \cdot \exp\left\{a_x \cdot (t - t_0)\right\} \tag{9}$$

Or, in terms of the original parameters of the dynamic system:

$$X(t) = \left[\frac{\varepsilon_k \cdot (1 - a_{11}) + \varepsilon_\lambda \cdot a_{22}}{A \cdot (1 - a_{11}) - D \cdot a_{22}}\right] +$$

$$+ \left[X(t_0) - \frac{\varepsilon_k \cdot (1 - a_{11}) + \varepsilon_\lambda \cdot a_{22}}{A \cdot (1 - a_{11}) - D \cdot a_{22}}\right] \cdot \exp\left\{(A \cdot (1 - a_{11}) - D \cdot a_{22})\,(t - t_0)\right\} \tag{10}$$

Once we know the fixed value of every parameter and the initial ones of the variables, the above expression determines what value corresponds to the instrumental variable $X(t)$ at any moment in time.

Given that, in a second step we will transform the initial non-linear system of two equations into a more general system, including the linear autonomous non-homogeneous differential equation with constant coefficients (8) and two linear differential equations with variable coefficients, which are directly deduced from equations (1) and (2) when we make some substitutions using both relations $K(t)^{a_{11}}\lambda(t)^{a_{22}} = K(t)/X(t)$ and $K(t)^{a_{11}-1}\lambda(t)^{1+a_{22}} = \lambda(t)/X(t)$. So, the dynamic equations for the two primary variables are:

$$\overset{\bullet}{K}(t) = \left(A - \frac{\varepsilon_k}{X(t)}\right) \cdot K(t) \tag{11}$$

$$\overset{\bullet}{\lambda}(t) = \left(D + \frac{\varepsilon_\lambda}{X(t)}\right) \cdot \lambda(t) \tag{12}$$

These two separated, non-autonomous but homogeneous, linear differential equations are connected by the simultaneous presence of $X(t)$ in their corresponding variable coefficients. The general expressions for a particular solution are respectively:

$$K(t) = K(t_0) \cdot \exp\left\{\int_{t_0}^{t} (A - \frac{\varepsilon_k}{X(s)}) \cdot ds\right\} \tag{13}$$

$$\lambda(t) = \lambda(t_0) \cdot \exp\left\{\int_{t_0}^{t} (D + \frac{\varepsilon_\lambda}{X(s)}) \cdot ds\right\} \tag{14}$$

Finally, the third step consists in determining the initial value of the costate variable $\lambda(t)$, or even a set of such values, for which trajectories become optimal (i.e. non-explosive). Given $K(t_0)$ known, this may be done by determining $X(t_0)$. In any case, once we determine $\lambda(t_0)$ equation (7) give us an initial value for $X(t)$, and by (9) we automatically get the whole stream of this instrumental variable. Then, using equations (13) and (14), we can determine the exact position of both $K(t)$ and $\lambda(t)$ at any moment in time. All the information that we need in this step will be supplied by the transversality condition. This necessary condition takes the following form:

$$\lim_{t\to\infty} \lambda(t_0)K(t_0) \cdot \exp\left\{-\frac{a_x}{b_x}\left(\varepsilon_k - \varepsilon_\lambda\right)(t - t_0)\right\} \cdot \exp\left\{(A + D - \theta)(t - t_0)\right\} \cdot$$

$$\cdot \left|\frac{b_x}{a_x X(t_0)} + \left(1 - \frac{b_x}{a_x X(t_0)}\right) \cdot \exp\left\{a_x \cdot (t - t_0)\right\}\right|^{\frac{\varepsilon_k - \varepsilon_\lambda}{b_x}} = 0 \qquad (15)$$

Using the constraints on parameters that we have introduced in equations (5) and (6), which allow us to connect with a particular class of economic models, we find that $b_x = \varepsilon_k - \varepsilon_\lambda > 0$ and $D = \theta - A < 0$. Then, the transversality condition can be reduced to:

$$\lim_{t\to\infty} \left|\frac{b_x \cdot \exp\left\{-a_x(t - t_0)\right\}}{a_x X(t_0)} + 1 - \frac{b_x}{a_x X(t_0)}\right| = 0 \qquad (16)$$

Proposition 1 : *Given $b_x > 0$, the above optimality condition holds if, and only if, $a_x > 0$ and $K(t_0)^{1-a_{11}}\lambda(t_0)^{-a_{22}} = X(t_0) = \frac{b_x}{a_x}$. Consequently, along the optimal (non-explosive) trajectories, $X(t)$ will remain constant and equal to its initial stationary value $X(t_0)$.*

Proof: See Appendix A.

Proposition 2 : *The non-explosive solution trajectories for the variables involved in the modified Hamiltonian dynamic system considered in equations (1)-(6), are unique and may be characterized by the following set of equations:*

$$K(t) = K(t_0) \cdot \exp\left\{(A - \frac{\varepsilon_k}{b_x/a_x}) \cdot (t - t_0)\right\} \qquad (17)$$

$$\lambda(t) = \lambda(t_0) \cdot \exp\left\{(D + \frac{\varepsilon_\lambda}{b_x/a_x}) \cdot (t - t_0)\right\} \qquad (18)$$

$$X(t) = \frac{b_x}{a_x} \tag{19}$$

$$a_x \equiv A \cdot (1 - a_{11}) - D \cdot a_{22} > 0 \tag{20}$$

$$b_x \equiv \varepsilon_k \cdot (1 - a_{11}) + \varepsilon_\lambda \cdot a_{22} = \varepsilon_k - \varepsilon_\lambda > 0 \tag{21}$$

$$X(t_0) \equiv \lambda(t_0)^{-a_{22}} K(t_0)^{1-a_{11}} = \frac{b_x}{a_x} \tag{22}$$

$$K(t_0) = K_0 \tag{23}$$

Proof: When we assume the outcome given in Proposition 1, these expressions come in a natural way from the method in three steps just developed. Unicity, in turn, comes from the general instability identified by means of the transversality condition, summarized here in equations (20)-(23).

These trajectories may be seen as the optimal paths arising from an intertemporal optimization problem like (P). Looking at these equations we can identify what pattern of behavior could follow the variables of the model. Equations (22) and (23) determine the initial values for the three variables $K(t)$, $\lambda(t)$ and $X(t)$. Once these initial values have been specified, equations (17), (18) and (19) determine unequivocally which is the position of these variables at any moment in time. Equations (20), (21) and (22), in turn, constitute the set of constraints on parameters that must hold to guarantee non-explosivity. These constraints outline the relevant manifold, center or unstable depending on the case, as an hyperbolic shaped manifold in $\Re^2$.

According to equation (19) we find that $X(t)$ will remain constant. In general, using equations (13), (14) and (7) we can derive an alternative expression for this instrumental variable:

$$\lambda(t)^{-a_{22}} K(t)^{1-a_{11}} = X(t_0) \cdot \exp\left\{ \int_{t_0}^{t} (a_x - \frac{b_x}{X(s)}) \cdot ds \right\} \tag{24}$$

Then, substituting the results from equations (22) and (19), we get:

$$\lambda(t)^{-a_{22}} K(t)^{1-a_{11}} = \frac{b_x}{a_x} \tag{25}$$

This expression represents the standard equation for a rectangular hyperbola and establishes that when it holds, trajectories remain always fixed or moving on the same hyperbolic space.

In the course of the above paragraphs we have ensured non-explosivity assuming that $a_x > 0$, $b_x > 0$ and $X(t_0) = \frac{b_x}{a_x}$. However, in Appendix A we have largely studied the transversality condition, concluding that there are two sets of parameter values for which non-explosivity appears as a property of some trajectories. Because of its interest from the point of view of economics, in the next section we are going to analyze the dynamic implications of the first set, while the consequences for dynamics when $a_x < 0$ and $b_x < 0$, for any finite $X(t_0)$, will be analyzed in Appendix D.

4 Steady State, Optimality and Growth

In this section we study whether or not the dynamic system admits the existence of a steady state, as well as its unicity and stability. In order to do that, we start by characterizing the phase space. Then, for each one of the plausible situations, we analyze the properties of the optimal trajectories. Finally, we make our study extensive to those non-optimal paths which imply a sustained positive growth for the state variable $K(t)$.

First, consider the differential equation (1) and rearrange terms to have:

$$\dot{K}(t) = \left(A - \varepsilon_k K(t)^{a_{11}-1} \lambda(t)^{a_{22}}\right) \cdot K(t) \tag{26}$$

The set of points where the condition $\dot{K} = 0$ holds may be summarized by the following equation representing a rectangular hyperbola:

$$\lambda(t)^{-a_{22}} K(t)^{1-a_{11}} = \frac{\varepsilon_k}{A} \text{ , with } K(t), \lambda(t) \neq 0 \tag{27}$$

The alternative set of points located on the λ-axis where $K(t) = 0$ cannot be considered as a second solution for $\dot{K} = 0$ because in such a case an indetermination appears when we try to determine the value of the term $\varepsilon_k / K(t)^{1-a_{11}} \lambda(t)^{-a_{22}}$.

Consider now the second differential equation (2) and rearrange terms to have:

$$\dot{\lambda}(t) = \left(D + \varepsilon_\lambda K(t)^{a_{11}-1} \lambda(t)^{a_{22}}\right) \cdot \lambda(t) \tag{28}$$

The set of points where the condition $\dot{\lambda} = 0$ holds may be summarized by the following equation representing another rectangular hyperbola:

$$\lambda(t)^{-a_{22}} K(t)^{1-a_{11}} = -\frac{\varepsilon_\lambda}{D} \text{ , with } \lambda(t), K(t) \neq 0 \tag{29}$$

Once again, the apparent second solution for $\dot{\lambda}=0$, formed by the set of points located on the K-axis where $\lambda(t)=0$, cannot be taken into account because of the indetermination associated with the term $\varepsilon_\lambda / K(t)^{1-a_{11}} \lambda(t)^{-a_{22}}$.

Given the preceding analysis of the phase lines associated to the spaces $\dot{K}=0$ and $\dot{\lambda}=0$, if we combine equations (27) and (29) it is easy to deduce what is happening in terms of the existence of a steady state and its dynamic properties. On the one hand, we find that a continuum of steady states does exist when $\frac{\varepsilon_k}{A}=\frac{\varepsilon_\lambda}{-D}$, while the steady state 'strictu sensu' does not exist when $\frac{\varepsilon_\lambda}{-D} \lessgtr \frac{\varepsilon_k}{A}$. So, we can identify three different situations depending on the sign of the above relation among parameters.

Proposition 3 : *When the parameters in the dynamic system satisfy*

$$\frac{\varepsilon_\lambda}{-D}=\frac{\varepsilon_k}{A}=\frac{b_x}{a_x} \tag{30}$$

a multiplicity of steady states exists and the optimal (non-explosive) paths may be characterized in the following way: $X(t)=\lambda(t)^{-a_{22}} K(t)^{1-a_{11}}=\frac{b_x}{a_x}$, $K(t)=K(t_0)=K_0$, *and* $\lambda(t)=\lambda(t_0)$; $\forall t \geq t_0$, $a_x>0$, $b_x>0$.

Proof: Given Proposition 1, this is an immediate consequence of the substitution of equation (30) into equations (13) and (14), or into (17) and (18). The state and costate variables remain constant because the coefficients $A-\frac{\varepsilon_k}{X(t)}=A-\frac{\varepsilon_k}{b_x/a_x}$ and $D+\frac{\varepsilon_\lambda}{X(t)}=D+\frac{\varepsilon_\lambda}{b_x/a_x}$ are both zero.

In this case, the initial value for $\lambda(t)$ must satisfy the following equality constraint:

$$\lambda(t_0)=\left(\frac{b_x}{a_x} \frac{1}{K(t_0)^{1-a_{11}}}\right)^{\frac{1}{-a_{22}}} \tag{31}$$

Given a starting point, when it corresponds to one of the infinite multiple steady states, the system stands at that point forever.

On the other hand, when $X(t_0)>b_x/a_x$, given $b_x>0$ and $a_x>0$, the instrumental variable increases monotonically $\forall t \geq t_0$ according to:

$$X(t)=\lambda(t)^{-a_{22}} K(t)^{1-a_{11}}=X(t_0) \cdot \Psi(t)>\frac{b_x}{a_x} \tag{32}$$

The function $\Psi(t)$ is an increasing function of t. For that reason, this equation represents a whole family of rectangular hyperbolic curves. When it

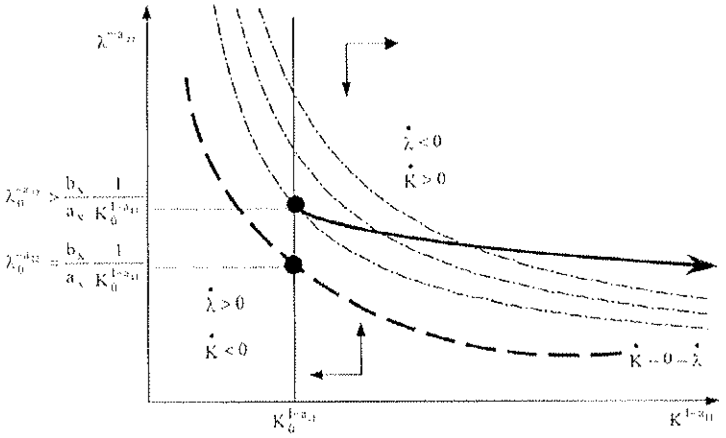

Figure 1: phase space corresponding to Proposition 3.

holds, trajectories that are non-optimal because the transversality condition breaks, pass continuously across different hyperbolic spaces from inside to outside. In this case we find that the state variable follows (13), but now the variable coefficient $A - \frac{\varepsilon_k}{X(t)}$ is always positive and higher than $A - \frac{\varepsilon_k}{b_x/a_x} = 0$, monotonically increasing and, in the long run, tends to $A > A - \frac{\varepsilon_k}{b_x/a_x} = 0$. Moreover, the costate variable follows (14), but now the variable coefficient $D + \frac{\varepsilon_\lambda}{X(t)}$ is always negative and lower than $D + \frac{\varepsilon_\lambda}{b_x/a_x} = 0$, monotonically decreasing and, in the long run, tends to $D < D + \frac{\varepsilon_\lambda}{b_x/a_x} = 0$. The initial value for $\lambda(t)$ could be any one which satisfies the following inequality constraint:

$$\lambda(t_0) > \left(\frac{b_x}{a_x}\frac{1}{K(t_0)^{1-a_{11}}}\right)^{\frac{1}{-a_{22}}} \tag{33}$$

Proposition 4 : *When the parameters in the dynamic system satisfy*

$$\frac{\varepsilon_\lambda}{-D} < \frac{\varepsilon_k}{A} < \frac{b_x}{a_x} \tag{34}$$

a steady state does not exist. However, the optimal (non-explosive) paths may be characterized by equations (17)-(23), with the particularity that the *constant coefficient* $A - \frac{\varepsilon_k}{b_x/a_x}$ *is positive and lower than* A*, while the constant coefficient* $D + \frac{\varepsilon_\lambda}{b_x/a_x}$ *is negative and higher than* D*. Moreover, given that* $X(t_0) = \frac{b_x}{a_x}$*, the initial value for* $\lambda(t)$ *satisfies the equality constraint (31).*

Proof: See Proposition 2, and then substitute the inequalities from (34) into the coefficients of equations (17) and (18).

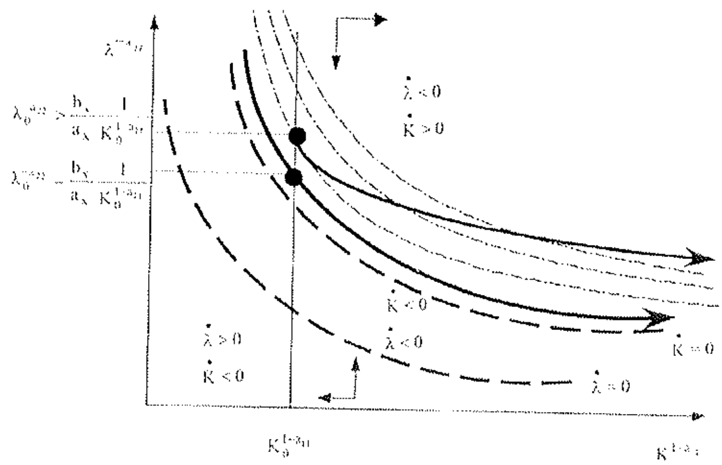

Figure 2: phase space corresponding to Proposition 4.

When the starting point is located on the rectangular hyperbola $X(t_0) = \frac{b_x}{a_x}$, the system remains forever on that space moving in the direction that implies a constant positive growth for the state variable.

On the other hand, when $X(t_0) > b_x/a_x$, given $b_x > 0$ and $a_x > 0$, the instrumental variable increases monotonically $\forall t \geq t_0$ according to (32). Trajectories, which are non-optimal because the transversality condition does not hold, move across different hyperbolic spaces from inside to outside. In this case the state variable follows (13) and the variable coefficient $A - \frac{\varepsilon_k}{X(t)}$ is always positive and higher than $A - \frac{\varepsilon_k}{b_x/a_x} > 0$, monotonically increasing and, in the long run, tends to $A > A - \frac{\varepsilon_k}{b_x/a_x} > 0$. Moreover, the costate variable follows (14) and the variable coefficient $D + \frac{\varepsilon_\lambda}{X(t)}$ is always negative and lower than $D+\frac{\varepsilon_\lambda}{b_x/a_x} < 0$, monotonically decreasing and, in the long run, tends to $D < D+\frac{\varepsilon_\lambda}{b_x/a_x} < 0$. The initial value for $\lambda(t)$ could be any one which satisfies the inequality constraint (33).

Proposition 5 *: When the parameters in the dynamic system satisfy*

$$\frac{\varepsilon_\lambda}{-D} > \frac{\varepsilon_k}{A} > \frac{b_x}{a_x} \tag{35}$$

a steady state does not exist. However, the optimal (non-explosive) paths may be characterized by equations (17)-(23), with the particularity that the constant coefficient $A - \frac{\varepsilon_k}{b_x/a_x}$ *is negative, while the constant coefficient* $D + \frac{\varepsilon_\lambda}{b_x/a_x}$ *is positive. Moreover, given that* $X(t_0) = \frac{b_x}{a_x}$*, the initial value for* $\lambda(t)$ *satisfies the equality constraint (31).*

Proof: See Proposition 2, and then substitute the inequalities from (35) in the coefficients of equations (17) and (18).

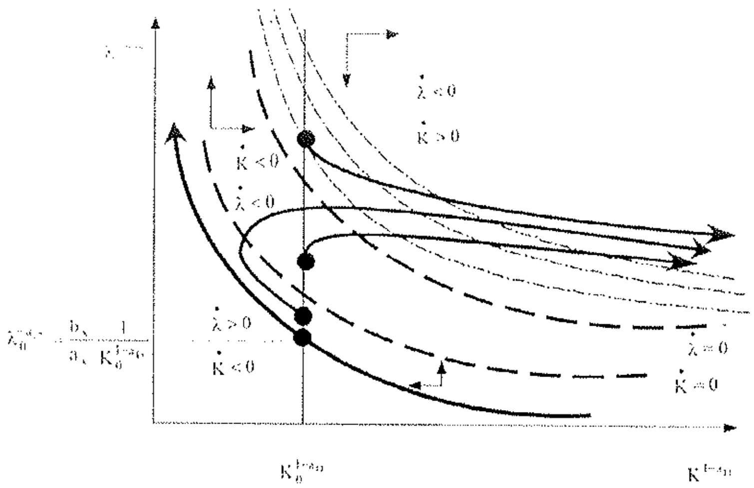

Figure 3: phase space corresponding to Proposition 5.

When the starting point is located on the rectangular hyperbola $X(t_0) = \frac{b_x}{a_x}$, the system remains forever on that space moving in the direction that implies a constant negative growth for the state variable.

On the other hand, when $X(t_0) > b_x/a_x$, given $b_x > 0$ and $a_x > 0$, the instrumental variable increases monotonically $\forall t \geq t_0$ according to (32), the transversality condition does not hold, and the corresponding non-optimal trajectories move continuously across different hyperbolic spaces from inside to outside. In this case, however, the state variable follows (13) but the variable coefficient $A - \frac{\varepsilon_k}{X(t)}$ is negative for $X(t) < \varepsilon_k/A$ and positive for $X(t) > \varepsilon_k/A$. In any case, this coefficient is higher than $A - \frac{\varepsilon_k}{b_x/a_x} < 0$, increases monotonically and, in the long run, tends to $A > A - \frac{\varepsilon_k}{b_x/a_x} < 0$. The costate variable, in turn, follows (14) but the variable coefficient $D + \frac{\varepsilon_\lambda}{X(t)}$ is positive for $X(t) < \varepsilon_k/A$ and negative for $X(t) > \varepsilon_k/A$. In any case, this coefficient is lower than $D + \frac{\varepsilon_\lambda}{b_x/a_x} > 0$, decreases monotonically and, in the long run, tends to $D < D + \frac{\varepsilon_\lambda}{b_x/a_x} > 0$. The initial value for $\lambda(t)$ could be any one which satisfies the inequality constraint (33).

5 Conclusions

The lack of a general solution method for non-linear dynamic systems is well known. The MHDS that are widely used in economics, constitute a clear example of such dynamic systems when either the return function and/or the dynamic constraints are non-linear. In this paper we have provided an analytical method for solving in closed form a general class of MHDS. The method works in three steps and, basically, consists in introducing an instrumental variable which allows to transform the original non-linear system into a system of separated linear differential equations.

The relevance of this method comes from the fact that the original non-linear system is structurally unstable and, consequently, the linear approximation cannot be used to study its dynamic properties. The dynamic system has not a unique nor isolated fixed point equilibrium. Despite this, applying our method we can find out exactly what the position of the system is at any moment depending on its initial starting point. The availability of an exact solution for each variable allows us to avoid the usual alternative based upon the reduction of dimension, which entails a significant loss of information. Moreover, indirectly we have shown how to analyze the dynamic properties of a system in presence of a continuum of steady states, and in absence of steady state as well.

In this context, the search for non-explosive trajectories require that parameters accommodate to the values provided by the transversality condition. According to a first set of such values, the dynamic system may be characterized as globally asymptotically unstable. However, there is a unique hyperbolic shaped space where trajectories remain fixed forever or, alternatively, move along such a curve with constant growth rates for the variables. Any other trajectory, starting out from the relevant manifold, will eventually explode. The system does not offer any possibility for transitional dynamics in this case.

According to a second set of parameter values, the dynamic system is globally asymptotically stable in terms of the growth rates of the variables, but it could also be globally asymptotically stable in the levels of such variables. An important property of the dynamic system, in this case, is the existence of transitional dynamics. However, indeterminacy associated to the initial starting point exist, since there is a multiplicity of non-explosive trajectories satisfying simultaneously the equations of the system and its boundary conditions.

Our analytical method may be used to study standard intertemporal optimization problems such as endogenous growth models of the AK type, and real business cycle models with unbounded technological constraints. Future research will concentrate on these two kinds of economic problems. In particular, we are going to analyze the performance of the solution method in models with two controls and one state variable, where it seems that the method can appropriately be applied. Our insight is that when the model includes more controls than costate variables, given an objective functional that combines the CRRA and the Cobb-Douglas form as well as a technological constraint exhibiting constant returns to the single accumulable factor, then the method proposed in this article becomes very useful.

6 Appendix A: Transversality condition.

The transversality condition (4), when we substitute using equations (13) and (14), becomes:

$$\lim_{t\to\infty} \lambda(t_0)K(t_0)\cdot \exp\left\{-\left(\varepsilon_k-\varepsilon_\lambda\right)\int_{t_0}^{t}\frac{ds}{X(s)}\right\}\cdot \exp\left\{\left(A+D-\theta\right)\left(t-t_0\right)\right\}=0 \tag{A1}$$

Given that we know the particular structure of $X(t)$ from equation (9), we can solve the integral appearing in the above expression.

$$\int_{t_0}^{t} \frac{ds}{X(s)} = \int_{t_0}^{t} \frac{ds}{\left(\frac{b_x}{a_x} + \left[X(t_0) - \frac{b_x}{a_x}\right] \cdot \exp\left\{a_x \cdot (s - t_0)\right\}\right)} =$$

$$= \frac{a_x (t - t_0) + \left[\ln\left|-a_x X(t_0)\right| - \ln\left|-a_x X(t_0) \cdot e^{a_x (t-t_0)} + b_x \left(-1 + e^{a_x (t-t_0)}\right)\right|\right]}{b_x} \tag{A2}$$

Now, introducing this result into the transversality condition we get:

$$0 = \lim_{t \to \infty} \lambda(t_0) K(t_0) \cdot \exp\left\{(A + D - \theta)(t - t_0)\right\} \cdot$$

$$\cdot \exp\left\{\frac{(\varepsilon_k - \varepsilon_\lambda)\left[a_x t_0 - \ln\left|-a_x X(t_0)\right|\right]}{b_x}\right\} \cdot$$

$$\cdot \exp\left\{-\frac{(\varepsilon_k - \varepsilon_\lambda)\left[a_x t - \ln\left|-a_x X(t_0) \cdot e^{a_x (t-t_0)} + b_x \left(-1 + e^{a_x (t-t_0)}\right)\right|\right]}{b_x}\right\} \tag{A3}$$

Rearranging terms we have:

$$0 = \lim_{t \to \infty} \lambda(t_0) K(t_0) \cdot \exp\left\{-\left(\frac{a_x (\varepsilon_k - \varepsilon_\lambda)}{b_x}\right)(t - t_0)\right\} \cdot \exp\left\{(A + D - \theta)(t - t_0)\right\} \cdot$$

$$\cdot \exp\left\{\frac{(\varepsilon_k - \varepsilon_\lambda)}{b_x}\left[-\ln\left|-a_x X(t_0)\right| + \ln\left|-a_x X(t_0) \cdot e^{a_x (t-t_0)} - b_x + b_x \cdot e^{a_x (t-t_0)}\right|\right]\right\} \tag{A4}$$

Take momentarily the last exponential term and simplify in the following way:

$$\exp\left\{\left(\frac{\varepsilon_k - \varepsilon_\lambda}{b_x}\right)\left[\ln\left|\frac{1}{-a_x X(t_0)}\right| + \ln\left|-a_x X(t_0) \cdot e^{a_x (t-t_0)} - b_x + b_x \cdot e^{a_x (t-t_0)}\right|\right]\right\} =$$

$$= \exp\left\{\left(\frac{\varepsilon_k - \varepsilon_\lambda}{b_x}\right) \ln\left|\left(1 - \frac{b_x}{a_x X(t_0)}\right) \cdot e^{a_x (t-t_0)} + \frac{b_x}{a_x X(t_0)}\right|\right\} =$$

$$= \left| \frac{b_x}{a_x X(t_0)} + \left(1 - \frac{b_x}{a_x X(t_0)}\right) \cdot \exp\left\{a_x \cdot (t - t_0)\right\} \right|^{\frac{\varepsilon_k - \varepsilon_\lambda}{b_x}} \tag{A5}$$

Now, coming back to the transversality condition we find an expression which corresponds to equation (15) in the main text:

$$\lim_{t\to\infty} \lambda(t_0) K(t_0) \cdot \exp\left\{ - \left(\frac{a_x (\varepsilon_k - \varepsilon_\lambda)}{b_x} \right) (t - t_0) \right\} \cdot \exp\left\{ (A + D - \theta)(t - t_0) \right\} \cdot$$

$$\cdot \left| \frac{b_x}{a_x X(t_0)} + \left(1 - \frac{b_x}{a_x X(t_0)}\right) \cdot \exp\left\{a_x \cdot (t - t_0)\right\} \right|^{\frac{\varepsilon_k - \varepsilon_\lambda}{b_x}} = 0 \tag{A6}$$

Given that initially we have assumed that $A + D - \theta = 0$ and $\varepsilon_k > \varepsilon_\lambda$, when we consider only finite initial values for both $\lambda(t)$ and $K(t)$, we can simplify to:

$$\lim_{t\to\infty} \exp\left\{ - \left(\frac{a_x (\varepsilon_k - \varepsilon_\lambda)}{b_x} \right) (t - t_0) \right\} \cdot$$

$$\cdot \lim_{t\to\infty} \left| \frac{b_x}{a_x X(t_0)} + \left(1 - \frac{b_x}{a_x X(t_0)}\right) \cdot \exp\left\{a_x \cdot (t - t_0)\right\} \right|^{\frac{\varepsilon_k - \varepsilon_\lambda}{b_x}} = 0 \tag{A7}$$

Finally, making some additional algebraic operations, the transversality condition may be written as follows:

$$\lim_{t\to\infty} \left| \frac{b_x \cdot \exp\left\{-a_x (t - t_0)\right\}}{a_x X(t_0)} + 1 - \frac{b_x}{a_x X(t_0)} \right|^{\frac{\varepsilon_k - \varepsilon_\lambda}{b_x}} = 0 \tag{A8}$$

This condition holds in both cases:
i) when $b_x > 0$, $a_x > 0$ and $X(t_0) = \frac{b_x}{a_x}$,
ii) when $b_x < 0$, $a_x < 0$ for any finite $X(t_0)$.

In the first case, taking equation (9) from the main text, we find that the instrumental variable remains constant, $X(t) = \frac{b_x}{a_x}$, $\forall t \geq t_0$. In the second case, the instrumental variable will converge to that value, from above or below, or it could even remain constant depending on its starting value.

7 Appendix B: Linearization of the original MHDS.

Consider the first order term of a Taylor-series expansion corresponding to the non-linear MHDS of equations (1) and (2), which may be written in matrix form as follows:

$$\begin{pmatrix} \overset{\bullet}{K}(t) \\ \overset{\bullet}{\lambda}(t) \end{pmatrix} = \begin{pmatrix} \Pi_k(K^*, \lambda^*) & \Pi_\lambda(K^*, \lambda^*) \\ \Gamma_k(K^*, \lambda^*) & \Gamma_\lambda(K^*, \lambda^*) \end{pmatrix} \cdot \begin{pmatrix} K(t) - K^* \\ \lambda(t) - \lambda^* \end{pmatrix} \qquad \text{(B1)}$$

In general, the coefficients of the Jacobian matrix take the form:

$$\Pi_k(K, \lambda) = -a_{11}\varepsilon_k K^{a_{11}-1}\lambda^{a_{22}} + A$$

$$\Pi_\lambda(K, \lambda) = -a_{22}\varepsilon_k K^{a_{11}}\lambda^{a_{22}-1}$$

$$\Gamma_k(K, \lambda) = (a_{11} - 1)\varepsilon_\lambda K^{a_{11}-2}\lambda^{1+a_{22}}$$

$$\Gamma_\lambda(K, \lambda) = (1 + a_{22})\varepsilon_\lambda K^{a_{11}-1}\lambda^{a_{22}} + D \qquad \text{(B2)}$$

The linearization we are considering here would make sense only when a steady state for the system exists. In such a case, as we can see in section 4, the result $\frac{\varepsilon_\lambda}{-D} = \frac{\varepsilon_k}{A}$ must hold, and this yields automatically a continuous multiplicity of steady states. Then, the previous coefficients evaluated at the steady state using the above result, take the form:

$$\Pi_k(K^*, \lambda^*) = (1 - a_{11})A$$

$$\Pi_\lambda(K^*, \lambda^*) = -a_{22}\varepsilon_k K^{*a_{11}}\lambda^{*a_{22}-1}$$

$$\Gamma_k(K^*, \lambda^*) = -(1 - a_{11})\varepsilon_\lambda K^{*a_{11}-2}\lambda^{*1+a_{22}}$$

$$\Gamma_\lambda(K^*, \lambda^*) = -a_{22}D \qquad \text{(B3)}$$

Now, calculating the determinant and the trace of the Jacobian matrix we get the following results:

$$det\ J^* \equiv \Pi_k^* \cdot \Gamma_\lambda^* - \Pi_\lambda^* \cdot \Gamma_k^* = 0$$

$$trace\ J^* \equiv \Pi_k^* + \Gamma_\lambda^* = \theta \tag{B4}$$

The result concerning the value of the trace is a standard result in optimal control theory when the objective functional is affected by a discount factor. However, the null determinant of the Jacobian matrix constitutes a major problem because of the non linearity of the original dynamic system. In this case the Hartman-Grobman theorem does not hold, the original system becomes structurally unstable in its steady state, and the linear approximation cannot be used to study the dynamic behavior of the original system either globally or locally.

8 Appendix C: Dynamics of the linear part of the original MHDS.

Consider the linear part of the non-linear MHDS of equations (1) and (2), which gives us the following dynamic system:

$$\overset{\bullet}{K}(t) = A \cdot K(t)$$

$$\overset{\bullet}{\lambda}(t) = D \cdot \lambda(t) \tag{C1}$$

The steady state, where $\overset{\bullet}{K}(t) = \overset{\bullet}{\lambda}(t) = 0$ takes place simultaneously, is unique and corresponds to the origin of coordinates, $(K^*, \lambda^*) = (0, 0)$. The Jacobian matrix of this linear autonomous and homogeneous system is:

$$J^* = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \tag{C2}$$

Its determinant is given by $det\ J^* = A \cdot D < 0$ and the trace, as usual, is $trace\ J^* = \theta > 0$. These features of the Jacobian matrix mean that the system has a saddle point dynamical structure. The two eigenvalues are respectively $\mu_1 = D < 0$ and $\mu_2 = A > 0$. Moreover, the two eigenvectors correspond to the canonical basis which span the plane. Consequently, the stable eigenspace coincides with the λ-axis and the unstable one coincides with the K-axis.

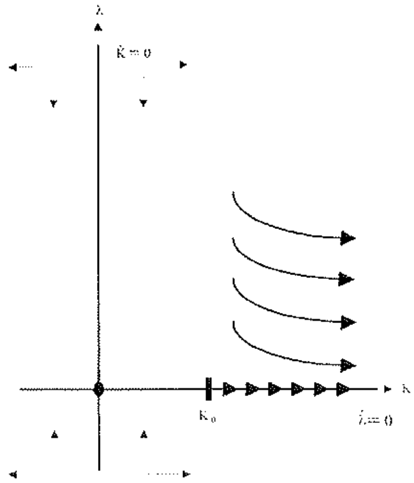

Figure 4: phase space corresponding to the system C1.

When we consider the initial condition $K(t_0) = K_0 > 0$ as well as the transversality condition $\lim_{t \to \infty} \lambda(t)K(t)\exp\{-\theta(t-t_0)\} = 0$, the above linear dynamic system has got a particular solution given by:

$$K(t) = K_0 \cdot \exp\{A \cdot (t - t_0)\}$$

$$\lambda(t) = \lambda(t_0) = 0 \qquad \text{(C3)}$$

9 Appendix D: The dynamics of the system when a_x and b_x are both negative.

In Appendix A we have studied the transversality condition, concluding that there are two parameter configurations for which non-explosivity could be considered as a property of some trajectories. Throughout the text, due to economic motivations, we have only analyzed the dynamic implications of one of them. However, non-explosivity, and consequently optimality, will also be ensured whenever $a_x < 0$, $b_x < 0$ for any finite $X(t_0)$. In that case, the instrumental variable $X(t)$ will converge to the value $\lim_{t\to\infty} X(t) = \frac{b_x}{a_x} > 0$. The convergence will come from above or below, depending on whether its starting value $X(t_0)$ is, respectively, higher or lower than $\frac{b_x}{a_x}$. When the initial value is higher, $X(t)$ decreases monotonically. When the initial value is lower, $X(t)$ increases monotonically. In the particular case where $X(t_0) = \frac{b_x}{a_x}$, the variable $X(t)$ will remain constant $\forall t \geq t_0$.

In this case, the non-explosive trajectories arising from our MHDS may be characterized by the following set of equations:

$$K(t) = K(t_0) \cdot \exp\left\{\int_{t_0}^{t} (A - \frac{\varepsilon_k}{X(s)}) \cdot ds\right\} \tag{13}$$

$$\lambda(t) = \lambda(t_0) \cdot \exp\left\{\int_{t_0}^{t} (D + \frac{\varepsilon_\lambda}{X(s)}) \cdot ds\right\} \tag{14}$$

$$X(t) = \frac{b_x}{a_x} + \left[X(t_0) - \frac{b_x}{a_x}\right] \cdot \exp\{a_x \cdot (t - t_0)\} \tag{9}$$

$$a_x \equiv A \cdot (1 - a_{11}) - D \cdot a_{22} < 0 \tag{20'}$$

$$b_x \equiv \varepsilon_k \cdot (1 - a_{11}) + \varepsilon_\lambda \cdot a_{22} < 0 \tag{21'}$$

$$X(t_0) \equiv \lambda(t_0)^{-a_{22}} K(t_0)^{1-a_{11}} \gtreqless \frac{b_x}{a_x} \tag{22'}$$

$$K(t_0) = K_0 \tag{23}$$

Now, given the multiple-valued equation (22') and the fixed value for $K(t_0)$ from equation (23), the above equations determine three different contexts.

First, when $X(t_0) = \frac{b_x}{a_x}$. Then, equation (9) reduces to (19) implying that $X(t)$ remains constant forever: $X(t) = \lambda(t)^{-a_{22}} K(t)^{1-a_{11}} = \frac{b_x}{a_x}$. This expression represents a rectangular hyperbola and establishes that the trajectories for both $K(t)$ and $\lambda(t)$ may be fixed or, alternatively, they move along a specific hyperbolic space. In fact, the state variable conforms to equation (17), but the constant rate of growth $A - \frac{\varepsilon_k}{b_x/a_x}$ may be positive, zero or negative depending on whether $\frac{\varepsilon_k}{A} \lesseqgtr \frac{b_x}{a_x}$. The costate variable, in turn, conforms to equation (18). For the sake of simplicity, from now on we will assume $D < 0$. With this assumption, we implicitly ratify equation (6) but not equation (5). Hence, we are still reasoning almost at the same level of generality as previous to that assumption. However, now it is easy to deduce that the constant rate of growth $D + \frac{\varepsilon_\lambda}{b_x/a_x}$ may be negative, zero or positive depending on whether $\frac{\varepsilon_\lambda}{-D} \lesseqgtr \frac{b_x}{a_x}$.

Second, when $X(t_0) > \frac{b_x}{a_x}$. Then, $X(t)$ decreases monotonically according to:

$$X(t) = \lambda(t)^{-a_{22}} K(t)^{1-a_{11}} = X(t_0) \cdot \Upsilon_1(t) \geqslant \frac{b_x}{a_x} \tag{D1}$$

The function $\Upsilon_1(t)$ is a decreasing function of t. For that reason, this equation describes a complete family of rectangular hyperbolic curves. In this context trajectories go across different hyperbolic curves along the phase space, from outside to inside, approaching the limiting one represented by equation (19). The state variable follows equation (13) and the variable growth rate $A - \frac{\varepsilon_k}{X(t)}$ is higher than $A - \frac{\varepsilon_k}{b_x/a_x}$, decreases monotonically over time and, eventually, tends to $\lim_{t \to \infty} \left(A - \frac{\varepsilon_k}{X(t)}\right) = A - \frac{\varepsilon_k}{b_x/a_x}$. However, this rate of growth may be positive, zero or negative depending on whether $\frac{\varepsilon_k}{A} \lesseqgtr X(t) \geqslant \frac{b_x}{a_x}$, but at the limit it only depends on whether $\frac{\varepsilon_k}{A} \lesseqgtr \frac{b_x}{a_x}$. The costate variable, in turn, follows equation (14) and the variable growth rate $D + \frac{\varepsilon_\lambda}{X(t)}$ is lower than $D + \frac{\varepsilon_\lambda}{b_x/a_x}$, increases monotonically over time and, eventually, tends to $\lim_{t \to \infty} \left(D + \frac{\varepsilon_\lambda}{X(t)}\right) = D + \frac{\varepsilon_\lambda}{b_x/a_x}$. However this rate of growth may be negative, zero or positive depending on whether $\frac{\varepsilon_\lambda}{-D} \lesseqgtr X(t) \geqslant \frac{b_x}{a_x}$, but at the limit it only depends on whether $\frac{\varepsilon_\lambda}{-D} \lesseqgtr \frac{b_x}{a_x}$.

Third, when $X(t_0) < \frac{b_x}{a_x}$. Then $X(t)$ increases monotonically according to:

$$X(t) = \lambda(t)^{-a_{22}} K(t)^{1-a_{11}} = X(t_0) \cdot \Upsilon_2(t) \leqslant \frac{b_x}{a_x} \tag{D2}$$

The function $\Upsilon_2(t)$ is an increasing function of t. This equation also describes a complete family of rectangular hyperbolic curves. Now trajectories go across different hyperbolic curves along the phase space, from inside to outside, approaching the limiting one given in equation (19). The state variable follows equation (13) and the costate variable follows equation (14). The variable growth rate $A-\frac{\varepsilon_k}{X(t)}$ is lower than $A-\frac{\varepsilon_k}{b_x/a_x}$, increases monotonically over time and, eventually, tends to $\lim_{t\to\infty}\left(A-\frac{\varepsilon_k}{X(t)}\right)=A-\frac{\varepsilon_k}{b_x/a_x}$. This rate of growth may be positive, zero or negative depending on whether $\frac{\varepsilon_k}{A}\lesseqgtr X(t)\leqslant\frac{b_x}{a_x}$, but at the limit it only depends on whether $\frac{\varepsilon_k}{A}\lesseqgtr\frac{b_x}{a_x}$. On the other hand, the variable growth rate $D+\frac{\varepsilon_\lambda}{X(t)}$ is higher than $D+\frac{\varepsilon_\lambda}{b_x/a_x}$, decreases monotonically over time and, eventually, tends to $\lim_{t\to\infty}\left(D+\frac{\varepsilon_\lambda}{X(t)}\right)=D+\frac{\varepsilon_\lambda}{b_x/a_x}$. This rate of growth may be negative, zero or positive depending on whether $\frac{\varepsilon_\lambda}{-D}\lesseqgtr X(t)\leqslant\frac{b_x}{a_x}$, but at the limit it only depends on whether $\frac{\varepsilon_\lambda}{-D}\lesseqgtr\frac{b_x}{a_x}$.

Once we have finished with the above classification, we find that each of them may be combined with the three possibilities concerning the existence of a steady state. These situations were characterized in the main text by the following relationships between parameters: (i) $\frac{\varepsilon_\lambda}{-D}=\frac{\varepsilon_k}{A}$, (ii) $\frac{\varepsilon_\lambda}{-D}<\frac{\varepsilon_k}{A}$, and (iii) $\frac{\varepsilon_\lambda}{-D}>\frac{\varepsilon_k}{A}$. In the case of the first one we have a multiplicity of steady states, while in the case of the second and third ones a steady state does not exist. Given the level of generality at which we are analyzing the dynamic system, if we do not introduce the assumption issued in equation (5), it is not possible to establish unambiguously a relationship between $\frac{b_x}{a_x}$ and both $\frac{\varepsilon_\lambda}{-D}$ or $\frac{\varepsilon_k}{A}$. So, a complete analysis must include the study of the following cases: (i.1) $\frac{\varepsilon_\lambda}{-D}=\frac{\varepsilon_k}{A}<\frac{b_x}{a_x}$, (i.2) $\frac{\varepsilon_\lambda}{-D}=\frac{\varepsilon_k}{A}=\frac{b_x}{a_x}$, (i.3) $\frac{\varepsilon_\lambda}{-D}=\frac{\varepsilon_k}{A}>\frac{b_x}{a_x}$, (ii.1) $\frac{\varepsilon_\lambda}{-D}<\frac{\varepsilon_k}{A}<\frac{b_x}{a_x}$, (ii.2) $\frac{\varepsilon_\lambda}{-D}<\frac{\varepsilon_k}{A}=\frac{b_x}{a_x}$, (ii.3) $\frac{\varepsilon_\lambda}{-D}<\frac{b_x}{a_x}<\frac{\varepsilon_k}{A}$, (ii.4) $\frac{\varepsilon_\lambda}{-D}=\frac{b_x}{a_x}<\frac{\varepsilon_k}{A}$, (ii.5) $\frac{b_x}{a_x}<\frac{\varepsilon_\lambda}{-D}<\frac{\varepsilon_k}{A}$, (iii.1) $\frac{b_x}{a_x}>\frac{\varepsilon_\lambda}{-D}>\frac{\varepsilon_k}{A}$, (iii.2) $\frac{b_x}{a_x}=\frac{\varepsilon_\lambda}{-D}>\frac{\varepsilon_k}{A}$, (iii.3) $\frac{\varepsilon_\lambda}{-D}>\frac{b_x}{a_x}>\frac{\varepsilon_k}{A}$, (iii.4) $\frac{\varepsilon_\lambda}{-D}>\frac{\varepsilon_k}{A}=\frac{b_x}{a_x}$, and (iii.5) $\frac{\varepsilon_\lambda}{-D}>\frac{\varepsilon_k}{A}>\frac{b_x}{a_x}$.

Fortunately, for every case it is easy to sign the growth rates for $K(t)$ and $\lambda(t)$. In any case, these rates of growth are equal to, or in the long run tend to, respectively, $A-\frac{\varepsilon_k}{b_x/a_x}$ and $D+\frac{\varepsilon_\lambda}{b_x/a_x}$, for any positive and finite $K(t_0)$ and $\lambda(t_0)$. As a result, we can say that the dynamic system is, in terms of its growth rates, globally asymptotically stable. Furthermore, for certain particular cases like the previous (i.2), it could also be globally asymptotically stable in terms of the levels of the variables. Then, an interesting property of the dynamic system, when a_x and b_x are both negative, is that it produces transitional dynamics. However, it is also important to underline that in such a case we have indeterminacy due to the infinite non-explosive trajectories which satisfy simultaneously the equations of the system and its boundary conditions. Recently, Russell and Zecevic (2000) surveys the multiplicity of equilibria problem and gives an extensive list of references where the possibility of indeterminacy arises in models of economic growth. Even so, our analysis remains original and alternative because we avoid reducing dimension and exploit directly the closed form solution for the variables of the original system. In this way we show how to study this complex problem in situations characterized by the lack of a well-defined isolated steady state.

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