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ABSTRACT

This paper provides the complete closed-form solution to the Lucas two-sector model of endogenous growth. We study the issues of existence, uniqueness, multiplicity, positivity, transitional dynamics and long-run growth, related to the competitive equilibrium paths. We identify the parameter range where the different results hold and deduce the entire trajectories for the original variables. We revise the results on convergence and overtaking which arise from this model, and prove that the parameterization currently used as the background for an explanation of economic miracles and disasters, is not satisfactory because of its counterintuitive implications.

Keywords: Endogenous Growth, Closed Form Solution, Transitional Dynamics, Multiplicity, Overtaking, Convergence.

JEL classification: C61, C62, O41.

1 Introduction

In a recent paper, Easterly and Levine (2000) assert that empirical work in growth economics is still searching for an answer to the following questions: *i*) how a country like Argentina can go from being like the United States early in the twentieth century to the struggling middle-income country it is today? and *ii*) how a country like Korea or Thailand can go from being like Somalia to a country striving to be like the United States? These questions connect with the issue of the empirically proved existence of economic miracles and disasters, but they also point out that the Lucas' (1993) first inquiry into the matter of what is the best theoretical model allowing for such a result remains still unanswered.

Since the past decade many authors have worked on the mechanics of economic development having tried to supply a theory for economic miracles. It is well known, however, that the basic neoclassical theory which focuses on physical capital accumulation, cannot explain any of the stylized facts considered in the paper by Easterly and Levine nor the development miracle occurring in Korea and Taiwan but not in the Philippines[1]. On the other hand, Lucas (1993), after having analyzed different models in use, concludes that even though human capital accumulation has to be placed on the center of our inquiry, a model like his own two-sector model of endogenous growth with differentiated technologies producing human capital as complementary to physical capital [Lucas (1988)], is not capable of generating anything one could call a miracle[2]. In my opinion this model can perfectly account for most of the development facts appearing in the vast empirical growth literature. However, it is also true that any model viewing the growth miracles as exogenously driven productivity miracles cannot be a good description for reality. Undoubtedly, the most appropriate explanations are to come from the models going beyond factor accumulation, namely, models of total factor productivity which focus on technological change, adoption of new tech-

[1] Other interesting references which study some basic facts concerning the distribution of per capita income across countries are Parente and Prescott (1993) and Schmitz (1993).

[2] This model was first criticized on the basis of the human capital definition. Lucas (1993) and Schmitz (1993) pointed out that there are good reasons to refuse a description of knowledge accumulation through years of schooling because human capital can be accumulated in schools, in research organizations as well as at work in the course of producing goods and engaging in trade. Among them, learning on the job seems to be by far the most important one.

nologies, changes in the composition of production or externalities, as those surveyed by Jovanovic (1995) and Lucas (1993) among others. Nevertheless, in this paper I will come back to the Lucas (1988) model, which includes an externality associated with the human capital accumulation that affects the production of goods. Some authors have considered this model and its extensions as a valuable example of what might be a theory for economic miracles and disasters; particularly Benhabib and Perli (1994) and Xie (1994). These are the most basic among those developing transitional dynamics from an analytical point of view in the presence of the externality. It has to be noted, however, that the accumulation of human capital device is important for the endogenous growth result, but it is not fundamental for the issue of countries that eventually overtake another initially richer ones. Instead, the assumed externality turns out to be fundamental for the explanation of such growth experiences, usually classified as miracles or disasters. Both articles, and the whole literature in general, consider multiplicity and the associated indeterminacy as essential for a correct contextualization of the overtaking phenomenon but, as we will see below, this presumption deserves a little more scrutiny.

On one hand, Benhabib and Perli explore how indeterminacy can arise in the endogenous growth model of Lucas showing that it may be found within the range of parameter values that are empirically plausible. The study of indeterminacy carried out by these authors is a local one because they focus on the continuum of solution trajectories that exists in the neighborhood of a given balanced growth path. In fact, they work with a version of the Lucas model that has been reduced by one dimension, identifying the steady state(s) and investigating the stability properties of the Jacobian matrix on the basis of the implicit eigenvalues. On the other hand, Xie develops a method which may be called the explicit dynamics method. This one allows for a global stability analysis that is tackled under the parameter constraints which ensure the multiplicity of equilibrium paths. In his paper, Xie obtains closed form solutions which let him to produce crystal clear transitional dynamics. However, for this purpose he has to proceed under the simplifying assumption according to which the inverse of the intertemporal elasticity of substitution for consumption in utility function equals the elasticity of output with respect to physical capital in goods sector. There is a drawback of imposing such a restriction in terms of simulation exercises, but there is also a substantial reward in terms of searching for theoretical properties of the transitional dynamics.

In this paper we offer the complete closed-form solution to the Lucas two-sector model of endogenous growth. This new analytical solution is also developed under the assumption that the inverse of the intertemporal elasticity of substitution equals the physical capital share. In a Technical Appendix we provide, embedded in our own calculations, a revision in depth of the results from Xie (1994), Benhabib and Perli (1994) and Lucas (1988). We are, however, more exhaustive and precise because, on one hand, we identify more accurately the parameter range where the different results hold and, on the other hand, we deduce the entire trajectories for the original control, state and co-state variables. That is, we do not reduce the dimension of the original modified Hamiltonian dynamic system by rewriting its variables in ratios[3]. Our procedure takes the system in its original specification and then, after having found an integrable combination of state and co-state variables, solves by Bernoulli's method each differential equation in a sequential order. In a series of Propositions, Lemmas and Corollaries, we study the issues of existence, uniqueness, multiplicity and positivity, as well as transitional dynamics and long-run growth, for every variable and their competitive equilibrium solution trajectories. These results, developed in the Appendix, allow us to formulate a general Theorem in the main text which summarizes in terms of the solution trajectories for the level of output per capita. We derive exact expressions for both the equilibrium paths and the balanced growth paths, and we do that in two contexts: one where there exist multiplicity of solution trajectories and a second one where we only find uniqueness.

Along this paper we also prove how the most appealing parameterization of this economy, which gives rise to a multiplicity of solutions as well as indeterminacy, and that has been taken as the parameter sub-space where to look for an explanation of economic miracles and disasters, is not satisfactory because of its counterintuitive implications. Consequently, we will retain the alternative scenario, where the solution trajectory for each variable is unique,

[3]In this sense, we are coherent with what was postulated by Ruiz-Tamarit and Ventura-Marco (2001), who pointed out the dangers of reducing dimension when this technique is systematically used for analizing transitional dynamics in growth models. As we will see below, the procedure followed here allow us to deal with the levels of the original variables and their rates of growth. Therefore, we can connect with what really matter according to the standard convergence literature.

as the only realistic solution for the Lucas model. There, we study the issues of overtaking and convergence in both the levels and the rates of growth. Finally, we deduce the exact expressions for the speed of convergence and the rate of saving, which show the values of these variables in the short-run as well as in the long-run.

The next sections are organized as follows: Section 2 recovers the Lucas model in its standard formulation, giving the first order conditions which correspond to the modified Hamiltonian dynamic system that solves the general intertemporal optimization problem for a competitive economy. In a Technical Appendix we describe the solution method applied to this particular non-linear dynamic system, which gives in closed-form the solution trajectories for state, co-state and control variables. In Section 3, we analyze the economic results concerning growth, overtaking and convergence referred to the variable output per capita. Finally, Section 4 provides the main conclusions.

2 The Lucas model

Consider the Lucas (1988) two-sector endogenous growth model, with a production externality in the final good sector associated with the human capital accumulation, under its standardized formulation as it was studied in Xie (1994) and Benhabib and Perli (1994). We take into account a closed economy with competitive markets. The economy is populated with many identical, rational agents, facing up to the problem that consists in choosing the controls $c(t)$ and $u(t)$, $\forall t \geq 0$, which solve the following optimization problem:

$$Max \int_0^{\infty} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} N(t)\ e^{-\rho t}\ dt \tag{P}$$

subject to:

$$\dot{K}(t) = AK(t)^{\beta}[u(t)N(t)h(t)]^{1-\beta}h_a(t)^{\gamma} - N(t)c(t)$$

$$\dot{h}(t) = \delta[1-u(t)]h(t)$$

$$K(0) = K_0 > 0 \quad \text{and} \quad h(0) = h_0 > 0 \qquad \text{given.}$$

Here $c(t)$ is the stream of real per capita consumption of a single good. The instantaneous utility function is a CRRA function where σ represents the inverse of the intertemporal elasticity of substitution. The population at time t is $N(t)$, which is assumed to grow at a constant exogenously given rate λ. The constant ρ is the rate of time preference or discount rate. In this model $h(t)$ is the human capital level, or the skill level, of a representative worker while $u(t)$ is the fraction of non-leisure time devoted to goods production. The output, $Y(t)$, which may be allocated to consumption or to physical capital accumulation depends on the capital stock, $K(t)$, the effective work force, $u(t)N(t)h(t)$, and the average skill level of workers, $h_a(t)$. Parameter β is the elasticity of output with respect to physical capital, and γ is positive and intended to capture the external effects of human capital. In problem (P) the representative optimizing agent takes $h_a(t)$ as given and, consequently, the competitive solution will differ from the socially optimal allocation. The efficiency parameter A represents the constant technological level in the goods sector of this economy. It is assumed that the growth of human capital does not depend on the physical capital stock. It depends on the effort devoted to the accumulation of human capital, $1-u(t)$, as well as on the achieved human capital stock. The efficiency parameter δ represents the constant technological level in the educational sector. It also represents the maximal rate of growth for h attainable when all effort is devoted to human capital accumulation. Technology in goods sector shows constant returns to scale over private internal factors. Technology in educational sector is linear. For the sake of simplicity, it is assumed that there is no physical nor human capital depreciation. In this particular problem, we assume $\rho > \lambda$ and the constant intertemporal elasticity of substitution is allowed to be either $\frac{1}{\sigma} \lesseqgtr 1$. The current value Hamiltonian associated with the previous intertemporal optimization problem may be written as:

$$H^c(K, h, \theta_1, \theta_2, c, u; A, \sigma, \beta, \gamma, \delta, \{N(t), h_a(t) : t \geq 0\}) =$$

$$= \frac{c^{1-\sigma}-1}{1-\sigma} N + \theta_1 \left[A K^{\beta} (uNh)^{1-\beta} h_a^{\gamma} - Nc \right] + \theta_2 \delta (1-u) h, \qquad (1)$$

where θ_1 and θ_2 are the co-state variables for K and h, respectively. The term h_a, as we have seen, is taken as given in order to calculate the competitive equilibrium. Then, the set of equations arising from the necessary first order conditions, under the equilibrium condition $h_a = h$ which implies that all workers are treated as being identical, are:

$$c^{-\sigma} = \theta_1 \tag{2}$$

$$\theta_1 (1-\beta) AK^{\beta}(uNh)^{-\beta} Nh^{1+\gamma} = \theta_2 \delta h \tag{3}$$

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \theta_1 \beta AK^{\beta-1}(uNh)^{1-\beta} h^{\gamma} \tag{4}$$

$$\overset{\bullet}{\theta}_2 = \rho\theta_2 - \theta_1(1-\beta)AK^{\beta}(uN)^{1-\beta} h^{-\beta+\gamma} - \theta_2\delta(1-u) \tag{5}$$

$$\overset{\bullet}{K} = AK^{\beta}(uNh)^{1-\beta} h^{\gamma} - Nc \tag{6}$$

$$\overset{\bullet}{h} = \delta (1-u) h. \tag{7}$$

The boundary conditions include the initial conditions, $K(0) = K_0$ and $h(0) = h_0$, as well as the transversality conditions,

$$\lim_{t\to\infty} \theta_1 K \exp\{-\rho t\} = 0 \tag{8}$$

$$\lim_{t\to\infty} \theta_2 h \exp\{-\rho t\} = 0. \tag{9}$$

This completes the Lucas model. On the margin, according to (2), goods must be equally valuable in its two uses: consumption and physical capital accumulation; according to (3), time must be equally valuable in its two uses: production and human capital accumulation. Moreover, (4) and (5) are the usual intertemporal efficiency conditions for physical and human capital. Equations (6) and (7), in turn, represent their respective accumulation processes. In the Appendix the reader may find all these equations analyzed in a different way looking for an explicit closed-form solution for the states, co-states and controls. In the following section, however, we will supply the main results of the article.

3 Economic Results

In this section we will concentrate on the results concerning the trajectory solution for per capita production, under the simplifying assumption that the inverse of the intertemporal elasticity of substitution equals the physical capital share. Then, taking the results for controls, states and co-states derived in the Appendix, we can formulate the following theorem:

Theorem 1 *Under the equilibrium conditions:*

I) if the externality associated with the human capital stock is strong enough, $\left(\frac{\gamma-\beta}{\beta}\right)\rho > \delta\left(1+\gamma-\beta\right)-\rho > 0$, $\frac{\rho-\delta}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$ *and* $\rho > \delta$ *then there exist a continuum of equilibrium paths for per capita production, starting from* $y_0 = \left[1+\Delta\right]^{1-\beta} Ak_0^{\beta}h_0^{1+\gamma-\beta}\left(\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\gamma-\beta)}\right)^{1-\beta}$. *Each of these paths may be characterized by the indeterminate value of the parameter* Δ *and takes only positive values. Moreover, all of them approach asymptotically to an exponential monotonic path which also depends on the value of* Δ, *and along which per capita production grows permanently at a positive constant rate,* $\bar{g}_y^I = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)$. *This implies that there exist a continuum of undetermined positive balanced growth paths and the variable output per capita shows transitional dynamics along any of the multiple equilibrium paths.*

II) if the externality associated with the human capital stock is not too strong, $-\left(\frac{\beta-\gamma}{\beta}\right)\rho < \delta\left(1+\gamma-\beta\right)-\rho < 0$ *and* $\delta > \rho$ *then output per capita follows a unique and positive equilibrium path, starting from* $y_0 = Ak_0^{\beta}h_0^{1+\gamma-\beta}\left(-\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\beta-\gamma)}\right)^{1-\beta}$. *This trajectory approaches asymptotically to an exponential monotonic path along which per capita production grows permanently at a positive constant rate,* $\bar{g}_y^{II} = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)$. *This one constitutes the unique positive balanced growth path. Consequently, we find that the variable output per capita shows transitional dynamics along the unique equilibrium path.*

Proof of part I. A strong externality means that $\gamma > \beta$. Moreover, we have $\delta\left(1+\gamma-\beta\right)-\rho > 0$ and $\delta\beta\left(1+\gamma-\beta\right)-\gamma\rho < 0$ as well as $\rho > \delta$ and $\frac{\rho-\delta}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$. Hence, expressions for k, h and u taken from (46), (36) and (53) may be used to obtain the following expression for y:

$$y = Ak^{\beta}u^{1-\beta}h^{1+\gamma-\beta} = \frac{A\left(\frac{\beta}{\rho}\right)^{\beta} h_0^{1+\gamma-\beta}\left(\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\gamma-\beta)}\right)^{1-\beta}}{[1+\Delta]^{\frac{\beta(1+\gamma-\beta)}{\gamma-\beta}}\left[1-\frac{\Delta}{1+\Delta}\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{\gamma}{\gamma-\beta}}} \tag{10}$$

$$\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} + C_{\Delta}^0 h_0^{1+\gamma-\beta} I_{\Delta}(t)\right]^{\frac{\beta}{1-\beta}} \exp\left\{\left(\frac{\rho-\delta}{\gamma-\beta}-\delta\right)t\right\}.$$

The results concerning multiplicity and positivity are obvious given the previous parameter constraints. Transitional dynamics may be also checked given that each equilibrium trajectory in (10) approaches asymptotically to its associated positive balanced growth paths:

$$\bar{y}_I = \frac{A\left(\frac{\beta}{\rho}\right)^{\beta} h_0^{\frac{1+\gamma-\beta}{1-\beta}}\left(\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\gamma-\beta)}\right)^{1-\beta}}{[1+\Delta]^{\frac{\beta(1+\gamma-\beta)}{\gamma-\beta}}\left(-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta C_{\Delta}^0}\right)^{\frac{\beta}{1-\beta}}} \exp\left\{\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\}. \tag{11}$$

These trajectories, which represent the long-run per capita production level, show a direct dependence on h_0 as well as on the indeterminate value of the parameter Δ, but they are independent of k_0.

Proof of part II. A weak externality means that $\gamma < \beta$, and then the constraint $\Delta = 0$ applies together with $\delta(1+\gamma-\beta)-\rho < 0$, $\delta\beta(1+\gamma-\beta) - \gamma\rho > 0$ and $\delta > \rho$. Hence, expressions for h, k and u taken from (47), (39) and (55) determine the following expression for y:

$$y = Ak^{\beta}u^{1-\beta}h^{1+\gamma-\beta} = A\left(\frac{\beta}{\rho}\right)^{\beta} h_0^{1+\gamma-\beta}\left(-\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\beta-\gamma)}\right)^{1-\beta}$$

$$[\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} - \frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right] \exp\left\{\frac{(1-\beta)(\delta(1+\gamma-\beta)-\rho)}{(\beta-\gamma)\beta}t\right\} + \tag{12}$$

$$+\frac{(\beta-\gamma)\,\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\exp\left\{\frac{(1+\gamma-\beta)}{\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}]^{\frac{\beta}{1-\beta}}.$$

Consequently, the results concerning uniqueness and positivity are obvious given the previous parameter constraints. Moreover, the equilibrium trajectory in (12) approaches asymptotically to the unique positive balanced growth path:

$$\bar{y}_{II}=\frac{A\left(\frac{\beta}{\rho}\right)^{\beta}h_0^{\frac{1+\gamma-\beta}{1-\beta}}\left(-\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\beta-\gamma)}\right)^{1-\beta}}{\left(\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\beta C_0^0}\right)^{\frac{\beta}{1-\beta}}}\exp\left\{\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}, \quad (13)$$

what constitutes the proof of transitional dynamics. This asymptotic path, which gives the long-run per capita production levels, depends on h_0 but does not depend on k_0. □

The above Theorem divides the parameter space in two different and disjoint sets for which we prove that it does exist at least one solution trajectory to the non-linear dynamic system that solves the optimization problem (P) under competitive conditions. The first parameter configuration, corresponding to the case I, always implies a multiplicity of solutions. This multiplicity corresponds to what Benhabib and Perli (1994) call indeterminacy. It refers to the case of multiple solution trajectories starting from the same initial conditions for predetermined variables, all of them converging to a unique steady state or balanced growth path. It must be noted that this definition is different from the alternative one where multiplicity arises from the existence of multiple steady states for which the equilibrium solution trajectory is unique once the initial conditions are specified.

Taking into account what has been proved for the case I in Theorem 1 and the corresponding Propositions in the Appendix, we may conclude that in the short-run trajectories for the variables y, c, k, θ_1, h and u show transitional dynamics as well as convergence, in both the levels and the rates of growth, to their respective long-run trajectories or values. The previous conclusion also apply to the ratio between capital stocks and the relative prices associated with them. The exception is the shadow price of human capital θ_2

which, for any initial value, always grows exponentially at a constant positive rate with no transitional dynamics. Moreover, a special feature arises in this case because a country may choose its long-run balanced growth path and, consequently, the long-run levels of per capita production by deciding the indeterminate value of Δ in (11). As Benhabib and Perli (1994) pointed out, this characteristic allows to account for the diversity of growth experiences without any invocation to fixed effects, differences in exogenously determined policies or persistent country specific exogenous shocks because it is possible to view cultural and non-economic factors as affecting the value of Δ and, hence, the equilibrium trajectory, which may differ across countries during the transition as well as in the long-run. Once this parameter has been decided, given k_0 and h_0, the initial values $y(0)$ and $\bar{y}_I\ (0)$ are respectively determined by (10) and (11). However, even though the first one depends on the value k_0 in addition to h_0 and Δ, the second one only depends on h_0 and Δ. Instead, countries cannot influence the long-run rate of growth because this one appears determined by the usual technological and preference parameters, independently of the initial conditions and Δ.

Retaining these results, we can now examine what happens in terms of the traditional convergence hypothesis across countries. For this we will consider two different countries A and B. These two countries eventually converge in terms of rate of growth unless some fundamental difference affects their technological or preference parameters, but during the transition their growth rates may differ due to differences in the levels of both capital stocks and the effort devoted to human capital accumulation. However, things are very different in terms of levels. In this case there could be convergence or divergence, even overtaking, depending on the initial conditions and the value assigned to Δ, which in turn determines the initial value of u. First, if we consider that the two countries are equally endowed, $k_0^A = k_0^B$ and $h_0^A = h_0^B$, then the country with the highest value for Δ will experience higher levels of per capita production in the short-run, but the lowest levels in the long-run. That is, the country with an initially higher effort devoted to human capital accumulation starts below but will finish above. Second, consider the two countries endowed in a different way: $k_0^A > k_0^B$ and $h_0^A \leqq h_0^B$. In this case, if $\Delta^A \geqq \Delta^B$ and the initial current per capita production in country A is higher than in country B, there will be again overtaking in finite time because country B, which has a higher human capital stock, also makes a greater (or equal) effort in accumulating education. Although less likely, overtaking can still occurs if $\Delta^A < \Delta^B$ as long as $y^A(0) > y^B(0)$.[4] Third, pure divergence appears when we observe that $k_0^A \geqq k_0^B$, $h_0^A > h_0^B$ and $\Delta^A \gtreqless \Delta^B$ provided that $y^A(0) > y^B(0)$ and $\bar{y}_I^A\ (0) > \bar{y}_I^B\ (0)$. Finally, convergence in levels will be the result observed when $k_0^A \gtrless k_0^B$ combines with $h_0^A = h_0^B$ and $\Delta^A = \Delta^B$ causing $\bar{y}_I^A\ (0) = \bar{y}_I^B\ (0)$. Therefore, convergence in levels is only possible among countries when they share the same long-run growth path. In short, these cases show the greater relevance of human capital stock and the effort devoted to its accumulation face to physical capital stock, in determining the pattern of growth followed by different countries over time.

These results are very appealing from the point of view of the empirical evidence concerning the recent growth experience of some countries as described by Lucas (1993) and others. At first, it seems a good theory for economic miracles and disasters because there is convergence in the rates of growth but not necessarily in the levels. However, additional inspection into such results show immediately that this is not so.

In Theorem 1, under the parameter constraints corresponding to case I, any of the multiple equilibrium paths for output per capita implies a positive rate of growth that approaches asymptotically to the unique and positive long-run growth rate $\bar{g}_y^I = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)$, as given in (11). This rate of growth is also shared in the long-run by other variables such as physical capital stock and per capita consumption, as shown in Propositions 9 and 11. Moreover, for the same interval of parameter values, Proposition 5 says that any of the multiple equilibrium paths for human capital stock implies a positive rate of growth that approaches asymptotically to the unique and positive long-run growth rate $\bar{g}_h^I = \frac{\rho-\delta}{\gamma-\beta}$, as given in (38). Associated with this range of parameter values, we also find that the effort devoted to goods production follows one of the multiple paths which asymptotically approaches to the unique and positive value $\bar{u} = \frac{\delta(1+\gamma-\beta)-\rho}{\delta(\gamma-\beta)}$, as given in (54). Finally, according to Proposition 1, the shadow price of human capital experiences continuous and sustained growth at a constant rate: $\rho - \delta$, even in the short-run.

[4] We have shown two cases where overtaking occurs under certain favorable initial endowments. Xie (1994), in his Theorem 3, considers the more implausible, but still feasible, case where overtaking takes place if Δ^A is sufficiently larger than Δ^B, given $k_0^A > k_0^B$ and $h_0^A > h_0^B$. This extreme case contradicts the Lucas' (1988) conjecture according to which the country with a greater initial endowments of physical and human capital will be permanently richer than the one with lower initial endowments.

The rationale for my criticism is that this case, where $\gamma > \beta$ and $\rho > \delta$, is not in general a good description for an economy because of its strong and evident counterintuitive predictions about the following three aspects of the model economy:

i) the sign of the relationship between the value of the long-run growth rate and the parameters determining such a value;

ii) the sign of the relationship between the long-run value of u and those parameters; and

iii) the implied evolution for the human capital stock and its shadow price along the transition as well as in the long-run.

On the one hand, we find that the higher the positive externality parameter γ, the lower the rate of growth of per capita physical and human capital, consumption and income; the higher the impatience degree or discount rate ρ, the higher the economy's growth rate; and the higher the efficiency parameter in the educational sector δ, the lower the rate of growth. On the other hand, we find that the higher the externality parameter γ, the higher the fraction of non-leisure time devoted to goods production u; the higher the discount rate ρ, the lower the fraction u; and the higher the efficiency parameter δ, the lower the effort devoted to the accumulation of human capital $1 - u$. Finally, we also find that the shadow price of human capital grows in parallel to the positive growth of the human capital stock. Nonetheless, this simultaneous growth of price and quantity is still compatible with the transversality condition.[5]

All these results depict an unconvincing scenario and, therefore, we may conclude that the case I must not be used as a source of empirically testable hypotheses. It cannot be used to interpret growth facts as those showed by recent empirical literature either. However, when we leave out case I we also miss the opportunity for a multiplicity of equilibrium paths and, at the same time, the chance for a multiplicity of balanced growth paths. Nevertheless, we still have the case II as a useful tool looking for an interpretation of the recent

[5] Although we do not supply a formal proof here, it might be shown that these results are independent of the assumption $\sigma = \beta$. In fact, the very implausible dependences of the long-run rate of growth and the fraction of non-leisure time devoted to goods production with respect to the discount rate and the productivity parameter in educational sector are also present in Benhabib and Perli (1994) under the assumption $\sigma \neq \beta$. Moreover, if the intertemporal elasticity of substitution is greater than one we could even find the unlikely dependences with respect to the externality parameter.

growth experiences. Recall that case II applies under the following parameter constraints: $\gamma < \beta$, $\Delta = 0$, $\delta(1+\gamma-\beta)-\rho < 0$, $\delta\beta(1+\gamma-\beta)-\gamma\rho > 0$ and $\delta > \rho$. Consequently, from (12), the rate of growth of per capita production may be written as follows:

$$\frac{1}{y(t)}\frac{dy(t)}{dt} = \frac{\delta(1+\gamma-\beta)-\rho}{\beta-\gamma} + \frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(1-\beta)(\beta-\gamma)}\left[\frac{\left(\frac{\bar{k}}{h}\right)_{II}(t)}{\left(\frac{k}{h}\right)(t)}\right]^{1-\beta}. \quad (14)$$

Moreover, given the results in Propositions 8 and 11, the rate of growth of per capita physical capital stock, which is equal to the rate of growth of per capita consumption, may be written as follows:

$$\frac{1}{k(t)}\frac{dk(t)}{dt} = \frac{1}{c(t)}\frac{dc(t)}{dt} = -\frac{\rho}{\beta} + \frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(1-\beta)\beta(\beta-\gamma)}\left[\frac{\left(\frac{\bar{k}}{h}\right)_{II}(t)}{\left(\frac{k}{h}\right)(t)}\right]^{1-\beta}. \quad (15)$$

It is easy to prove that the previous rates of growth for y, c and k are above or below their common long-run growth rate $\bar{g}_y^{II} = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)$ depending on whether the ratio $\left(\frac{k}{h}\right)$ is below or above its long-run level $\left(\frac{\bar{k}}{h}\right)_{II}$. The latter, in turn, depends on whether $h_0 \gtreqqless \left(\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\beta C_0^0}\right)^{\frac{1}{1+\gamma-\beta}} \left(\frac{\rho}{\beta}\right)^{\frac{1-\beta}{1+\gamma-\beta}} k_0^{\frac{1-\beta}{1+\gamma-\beta}}$. Moreover, the long-run value of the rate of growth depends positively on the parameter associated with the productivity of human capital in educational sector, δ, as well as on the parameter associated with the technological externality, γ. It also depends negatively on the rate of discount, ρ. The effect of the aggregate physical capital share in goods sector, β, is ambiguous given the presence of the externality. According to the previous results, we can see that those rates of growth do not show the usual U-shape [Mulligan and Sala-i-Martín (1993)], given that there is a fundamental asymmetry between trajectories that flow across the space characterized by the inequality $\frac{\left(\frac{k}{h}\right)(t)}{\left(\frac{\bar{k}}{h}\right)_{II}(t)} < 1$ and those for which $\frac{\left(\frac{k}{h}\right)(t)}{\left(\frac{\bar{k}}{h}\right)_{II}(t)} > 1$. This is so because under the parameter constraints corresponding to the case II, variable $h(t)$ grows permanently at a constant rate and $u(t)$ remains constant forever along their respective balanced growth paths. In any case, the transitional rates of growth converge, from above or below, to their long-run values[6]. This convergence is a direct consequence of the convergence to unity experienced by the ratio:

$$\frac{\left(\frac{k}{h}\right)(t)}{\left(\frac{\bar{k}}{h}\right)_{II}(t)} = \tag{16}$$

$$\left(1+\left[\frac{\left(\frac{\rho}{\beta}k_0\right)^{1-\beta}\left(\delta\beta\left(1+\gamma-\beta\right)-\gamma\rho\right)}{\left(\beta-\gamma\right)\beta C_0^0 h_0^{1+\gamma-\beta}}-1\right]\exp\left\{-\frac{\delta\beta\left(1+\gamma-\beta\right)-\gamma\rho}{\beta\left(\beta-\gamma\right)}t\right\}\right)^{\frac{1}{1-\beta}}.$$

Taking this result together with what was proved for the case II in Theorem 1, we may conclude that in the short-run trajectories for the variables y, c and k show transitional dynamics as well as convergence to their respective long-run levels and rates of growth. The trajectory for the shadow price of physical capital stock, θ_1, also shows transitional dynamics and convergence in its level and rate of growth, as it is shown in the proof of Proposition 10. Moreover, we find that price and quantity evolve in opposite directions. On the other hand, we prove in Proposition 6 that the rate of growth for h is constant along the unique solution trajectory while, in Proposition 12, we prove that the rate of growth for u is also constant and equal to zero[7]. Strictly speaking, we find that there are no transitional dynamics of any kind for these two variables: their unique equilibrium paths coincide with their unique balanced growth paths. The same is true for θ_2, the shadow price of human capital stock, which decreases at a constant rate along the unique equilibrium trajectory, as it is shown in Proposition 1. In this case price and quantity also evolve in opposite directions. As a consequence of the previous statements, we find that the trajectories for the ratio between the stocks and the relative prices corresponding to them experience transitional dynamics

[6]Moreover, if we define a broad output in per capita terms as $q(t) = y(t) + \frac{\theta_2(t)}{\theta_1(t)}\overset{\bullet}{h}(t)$, which includes the value in units of goods of the gross investment in human capital then, given that $u(t)$ is constant, the rate of growth for $q(t)$ equals the rate of growth for $y(t)$.

[7]In fact, Proposition 12 proves that effort devoted to goods production is constant and equal to $u = -\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\beta-\gamma)}$. Hence, we deduce that the effort devoted to human capital accumulation depends positively on the productivity of human capital in educational sector δ, as well as on the weight of the technological external effect γ; but it depends negatively on the rate of discount ρ, as well as on the physical capital share in goods sector β.

and convergence in both their levels and rates of growth, according to what is shown in Propositions 14 and 15.

Now, taking two different countries A and B, we may inquire about the story which arises from the model in terms of the convergence hypothesis. Once again, we find that the two countries will converge in terms of rate of growth unless some fundamental difference affects their technological or preference parameters, but during the transition their growth rates may differ due to differences in the levels of both human and physical capital stocks. According to (14), the current rate of growth of per capita production depends positively on the distance between the current level and the level associated with the balanced growth path of the ratio between physical and human capital stocks. On the other hand, things are different in terms of the levels of output per capita since there could be overtaking, divergence without overtaking or convergence, depending on the initial conditions. First, consider the two countries endowed as follows: country A has a bigger initial physical capital stock, $k_0^A > k_0^B$, but a smaller initial human capital stock, $h_0^A < h_0^B$. Then, as long as the initial current per capita production in country A is still bigger than in country B, $y^A(0) > y^B(0)$, this one will overtake country A in a finite time period given that $\bar{y}_{II}^A(0) < \bar{y}_{II}^B(0)$. This possibility was not accepted by Xie (1994) who considered that under the constraints corresponding to case II no overtaking could happen. Here, overtaking appears associated with the differentiated endowment of capital stocks alone. Poor countries can not use arbitrarily their decision on effort devoted to human capital accumulation for influencing on their roads to growth, and so overtake the richer ones, because transversality condition imposes uniqueness with a fixed stationary value for u. This kind of overtaking is absolutely dependent on the initial endowment of human capital and is very suitable for explaining the after second world war experience of Japan and Germany[8]. The country with the highest stock of human capital will always emerge in finite time as the richest country and maintain its position as long as its human capital advantage is sustained. In general, the model may explain miracles and disasters but only those which are, in a certain way, predetermined and foreseeable given the initial distribution across countries of both human and physical

[8] Parente and Prescott (1991) did not accept these arguments and the model behind as a theory for economic miracles, based on the experience of East Germany relative to West Germany. Nevertheless, they did not take into account external factors as the enormous foreign direct investment which largely influenced on such a comparison.

capital stocks. Second, pure divergence without overtaking also appears in this model when $k_0^A \geqq k_0^B$ and $h_0^A > h_0^B$, given that in such a case we have $y^A(0) > y^B(0)$ and $\bar{y}_{II}^A(0) > \bar{y}_{II}^B(0)$.[9] Third, we will observe convergence in levels when the initial distribution of capital stocks corresponds to $k_0^A \gtrless k_0^B$ and $h_0^A = h_0^B$ which implies $y^A(0) \gtrless y^B(0)$ and $\bar{y}_{II}^A(0) = \bar{y}_{II}^B(0)$. In short, these cases show how much relevant is human capital over physical capital, in determining the pattern of growth followed by different countries over time.

Next, looking for a quantitative measure of the previously considered convergence processes, we will provide the results concerning two additional variables. First, from the definition of speed of convergence and using (12), (13), (64) and (65), we deduce the following result:

$$SC(t) \equiv \frac{-\frac{d}{dt}\left[\ln\left(\frac{y(t)}{\bar{y}_{II}(t)}\right)\right]}{\ln y(t) - \ln \bar{y}_{II}(t)} = \frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\beta}\left[\frac{\left(\frac{\bar{k}}{h}\right)_{II}(t)}{\left(\frac{k}{h}\right)(t)}\right]^{1-\beta}. \quad (17)$$

This equation gives the speed of convergence for the current per capita production level to the corresponding long-run level. That is, the speed of convergence approaches, from above or below, to the long-run value $\lim_{t\to\infty} SC(t) = \frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\beta} > 0$, depending on whether $\frac{\left(\frac{k}{h}\right)(t)}{\left(\frac{\bar{k}}{h}\right)_{II}(t)} \lesseqqgtr 1$. Nevertheless, in this model the current rate of growth of per capita production, given in (14) as $g_y(t) \equiv \frac{1}{y(t)}\frac{dy(t)}{dt}$, also converges to the long-run growth rate $\bar{g}_y^{II} = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)$. Therefore, we apply again the definition of speed of convergence to this variable and get:

$$sc_g(t) \equiv \frac{-\frac{d}{dt}\left[\ln\left(\frac{g(t)}{\bar{g}}\right)\right]}{\ln\frac{g(t)}{\bar{g}}} = \frac{\frac{1+\gamma-\beta}{\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)}{1+\frac{(1-\beta)(\delta(1+\gamma-\beta)-\rho)}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\left[\frac{\left(\frac{k}{h}\right)(t)}{\left(\frac{\bar{k}}{h}\right)_{II}(t)}\right]^{1-\beta}}. \quad (18)$$

[9] It is obvious that the country with a greater initial endowments of both physical and human capital will be permanently richer than the one with lower initial endowments. In this case, there is no possibility for reversing that result which corresponds to the original Lucas' (1988) conjecture, although it was formulated thinking of a very different context.

The speed of convergence associated with the rate of growth of per capita production approaches, from above or below, to the long-run value $\lim_{t\to\infty} sc_g(t) = \frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\beta} > 0$, depending on whether $\frac{\left(\frac{k}{h}\right)(t)}{\left(\frac{\bar{k}}{h}\right)_{II}(t)} \gtreqless 1$ because now $\delta\left(1+\gamma-\beta\right) - \rho < 0$. Not surprisingly, we find that $\lim_{t\to\infty} sc_g(t)$ equals $\lim_{t\to\infty} SC(t)$.

In general, the long-run value of the speed of convergence depends positively on the parameter associated with the productivity of human capital technology, δ, as well as on the parameter associated with the technological externality, γ. However, it depends negatively on the intertemporal rate of discount, ρ. The effect of the elasticity of output with respect to physical capital, β, is ambiguous given the presence of the externality. The explicit influence of σ, the inverse of the intertemporal elasticity of substitution, is not evident because of the assumption $\sigma = \beta$. The previous results are according to Ortigueira and Santos (1997) who pointed out that if technological externalities are present, then preferences parameters may affect the speed of convergence. However, their method of study which implies first a reduction of dimension and thereafter a linearization in the neighborhood of the so artificially generated steady state, leads to propose an expression for the speed of convergence that corresponds to our constant long-run value. That procedure ignores the fact that along the transition the speed of convergence may vary significantly, although in a monotonous way. Hence, the asymptotic speed of convergence may be irrelevant for describing the convergence process. Provided that we consider a country for which its current $\frac{k}{h}$ values approach the long-run ones from below, expressions (17) and (18) show that along the transition the speed of convergence for the levels decreases while the speed of convergence for the rates of growth increases[10].

Second, from the definition of the saving rate corresponding to this model, and using (52), (47), (12), (64) and (65), we deduce the following result:

[10]These results are in accordance with Parente and Prescott (1994) who find in its calibrations for USA and Japan that growth rates are lower the closer a country is to its balanced growth path, but the speed of convergence is higher the closer a country is to its balanced growth path. With respect to the variability of the speed of convergence and its evolution along the transition see also Rappaport (2000).

$$s(t) \equiv 1 - \frac{c(t)}{y(t)} = 1 - \frac{\rho}{\beta}\frac{k(t)}{y(t)} = 1 - \frac{(1-\beta)\,\rho\,(\beta-\gamma)}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\left[\frac{\left(\frac{k}{h}\right)(t)}{\left(\bar{\frac{k}{h}}\right)_{II}(t)}\right]^{1-\beta}. \quad (19)$$

The saving rate converges, from above or below, to its unique long-run value $0 < \lim_{t\to\infty} s(t) = \frac{(1+\gamma-\beta)\beta(\delta-\rho)}{\delta\beta(1+\gamma-\beta)-\gamma\rho} < 1$, depending on whether $\frac{\left(\frac{k}{h}\right)(t)}{\left(\bar{\frac{k}{h}}\right)_{II}(t)} \lesseqqgtr 1$. This long-run saving rate depends positively on the productivity of human capital technology, δ, as well as on the size of the externality parameter, γ. Moreover, it depends negatively on the discount rate, ρ, but the effect of the aggregate physical capital share, β, remains ambiguous given the presence of a technological externality. Actually, this sign will be positive or negative depending on whether $\beta \gtrless \frac{1+\gamma}{2}$, if and only if $\gamma \neq 0$. Otherwise, this latter parameter would not have any influence on the asymptotic saving rate. Finally, from the previous equation it is also possible to specify the trajectories for the ratios $\frac{c(t)}{y(t)}$ and $\frac{k(t)}{y(t)}$, as well as their long-run constant values.

4 Conclusions

In searching for a theory of economic miracles we have revisited the Lucas (1988) two-sector model of endogenous growth with differentiated technologies producing human capital as complementary to physical capital, and an externality associated with the human capital accumulation that affects the production of goods. The human capital accumulation mechanism is responsible for the endogenous growth and transitional dynamics results while the externality, when strong, causes multiplicity and indeterminacy in connection with the strong increasing returns in the aggregate production function. The model has been completely solved in closed form, deducing the entire trajectories for all the original control, state and co-state variables which give us both the transitional and the long-run dynamics associated with the competitive equilibrium. This model shows two different sets of parameter constraints for which an equilibrium solution to the non-linear dynamic system does exist. The first one, which gives rise to a multiplicity of solutions, has been taken as the reference context where to look for an explanation of economic miracles and disasters, but we have shown that this is not satisfactory because of its strong and evident counterintuitive implications for the long-run values of the rate of growth and the fraction of non-leisure time devoted to goods production. The bad predictions are of the kind that the higher the impatience level, the higher the economy's growth rate and the lower the goods production effort; also, the higher the efficiency parameter in educational sector, the lower the rate of growth and the lower the effort devoted to the accumulation of human capital; or even the surprising permanent positive growth of the shadow price of human capital in parallel to the evolution of the stock. It has to be noted that these results are independent of the assumption $\sigma = \beta$. Thus, our refusal of the case which generates multiplicity is solid and does not depend on the simplifying assumptions adopted in this paper.

Consequently, we only retain as a realistic solution for the Lucas model the alternative scenario where the solution trajectory for each variable is unique. This one, as we have proved, is rich enough and useful to generate a plausible explanation for many of the recent growth experiences. Short-run trajectories for variables like output, consumption, physical capital and its shadow price, show transitional dynamics as well as convergence to their respective long-run levels and rates of growth. Moreover, the rate of growth for human capital and its shadow price are constant along the unique solution trajectory, while the rate of growth for effort is also constant and equal to zero. There are no transitional dynamics for these variables: their unique equilibrium paths coincide with their unique balanced growth paths. Trajectories for the ratio between the two capital stocks as well as the relative prices corresponding to them, experience transitional dynamics and convergence in both their levels and rates of growth. In this context, both the long-run rate of growth and effort devoted to the accumulation of human capital depend negatively on the rate of discount but positively on the efficiency in educational sector and on the weight of the technological external effect.

We have also studied the issues of overtaking and convergence across countries in both the levels and the rates of growth. Convergence in rates of growth is guaranteed unless some fundamental difference across countries affects their technological or preference parameters. However, convergence in levels is not ensured because there could be divergence and overtaking depending on the initial conditions. We have considered in detail some particular cases which show how much relevant is human capital over physical capital in determining the effective pattern of growth associated with each country. The country with the highest initial stock of human capital will always emerge in finite time as the richest country and maintain its position as long as its human capital advantage is sustained. Therefore, we may conclude that the indeterminacy result is sufficient for explaining stylized facts like disparity in per capita income levels as well as overtaking, but it is not necessary.

Finally, we have deduced the explicit expressions for the speed of convergence and the rate of saving. The first one has been calculated for the levels and also for the rate of growth of per capita production. Along the transition both converge and, as countries develop, the speed of convergence for the levels decreases but the speed of convergence for the rates of growth increases. In the long-run both are equal and depend positively on the efficiency in educational sector and on the weight of the external effect. Moreover, preferences parameters affect the long-run speed of convergence which depends negatively on the rate of discount. The saving rate also shows transitional dynamics decreasing along the development path. This rate converges to a constant value, which depends positively on the educational efficiency as well as on the externality, but negatively on the rate of discount.

5 Appendix: The Complete Analytical Solution

Previous to any transformation of equations (2)-(9) we introduce, as in Benhabib and Perli (1994), the simplifying assumption of a constant normalized population. According to the Lucas and Xie notation, this corresponds to $\lambda = 0$ and $N(0) = 1$. Then, from (2) and (3) we get the next two control functions:

$$c = \theta_1^{-\frac{1}{\sigma}}, \tag{20}$$

$$u = \left(\frac{(1-\beta)\,A}{\delta}\right)^{\frac{1}{\beta}} \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\beta}} h^{\frac{\gamma}{\beta}-1}k. \tag{21}$$

After substituting the above expressions in (4)-(7), we obtain the following non-linear dynamic system:

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \xi\theta_1^{\frac{1}{\beta}}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}h^{\frac{\gamma}{\beta}} \tag{22}$$

$$\overset{\bullet}{\theta}_2 = -\left(\delta-\rho\right)\theta_2 \tag{23}$$

$$\overset{\bullet}{k} = \frac{\xi}{\beta}\theta_1^{\frac{1}{\beta}-1}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}kh^{\frac{\gamma}{\beta}} - \theta_1^{-\frac{1}{\sigma}} \tag{24}$$

$$\overset{\bullet}{h} = \delta h - \left(\frac{1-\beta}{\beta}\right)\xi\theta_1^{\frac{1}{\beta}}\theta_2^{-\frac{1}{\beta}}kh^{\frac{\gamma}{\beta}}, \tag{25}$$

where k represents the aggregate as well as per capita physical capital stock, and $\xi \equiv \frac{\beta\delta}{1-\beta}\left(\frac{(1-\beta)A}{\delta}\right)^{\frac{1}{\beta}}$. These equations, together with the initial conditions, $k(0) = k_0$ and $h(0) = h_0$, and the transversality conditions (8) and (9), determine the dynamics of the Lucas economic system in equilibrium over time.

Proposition 1 *Along any equilibrium path, θ_2 grows permanently at a constant rate: $-\left(\delta-\rho\right)$. Each of these paths, in turn, represents a balanced growth path for θ_2.*

Proof. From (23) we obtain $\overset{\bullet}{\theta}_2 / \theta_2$ constant. So then,

$$\theta_2 = \theta_2(0) \exp \{-(\delta - \rho) t\}, \tag{26}$$

where $\theta_2(0)$ still has to be determined. □

Consider now the instrumental variable x defined as[11]:

$$x \equiv \theta_1^{\frac{1}{\sigma}} k. \tag{27}$$

By totally differentiating this expression and substituting from (22) and (24) we get:

$$\overset{\bullet}{x} = \frac{1}{\sigma} \frac{\overset{\bullet}{\theta}_1}{\theta_1} x + \frac{\overset{\bullet}{k}}{k} x = \frac{\rho}{\sigma} x - \frac{\xi}{\sigma} \theta_1^{\frac{1}{\beta}-1} \theta_2^{-\left(\frac{1-\beta}{\beta}\right)} h^{\frac{\gamma}{\beta}} x + \frac{\xi}{\beta} \theta_1^{\frac{1}{\beta}-1} \theta_2^{-\left(\frac{1-\beta}{\beta}\right)} h^{\frac{\gamma}{\beta}} x - \frac{x}{\theta_1^{\frac{1}{\sigma}} k},$$

which under the assumption $\sigma = \beta$ transforms into the following non homogeneous first-order first-degree linear differential equation with constant coefficients:

$$\overset{\bullet}{x} = \frac{\rho}{\sigma} x - 1. \tag{28}$$

Now, given the initial condition $k(0) = k_0$ and the initial value $\theta_1(0)$, although by the moment unknown, we can generate an initial condition for x, namely $x(0) = \theta_1^{\frac{1}{\sigma}}(0) k_0$. Then, a particular solution to (28) will be given by the expression:

$$x = \frac{\sigma}{\rho} + \left[x(0) - \frac{\sigma}{\rho}\right] \exp\left\{\frac{\rho}{\sigma} t\right\}. \tag{29}$$

The unique non-explosive solution trajectory for x is the one that implies a constant value given by the initial condition just introduced, $x = \frac{\sigma}{\rho} = x(0) = \theta_1^{\frac{1}{\sigma}}(0) k_0$. However, this is not a necessary condition. Instead, what must be satisfied is the transversality condition (8). Therefore, inspecting into that condition, we can establish and prove the following:

Proposition 2 *Along any equilibrium path, x remains constant at the stationary value $x = \frac{\sigma}{\rho}$.*

[11] This variable corresponds to the inverse of the ratio consumption-physical capital stock used by Xie in his article.

Proof. From (27) and (29), under the assumption $\sigma = \beta$, we get:

$$\theta_1 k = x\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} = \frac{\sigma}{\rho}\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} + \left[x(0) - \frac{\sigma}{\rho}\right]\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\left\{\frac{\rho}{\sigma}t\right\}.$$

Then, the transversality condition (8) may be written as:

$$\lim_{t\to\infty} \theta_1 k \exp\{-\rho t\} = \lim_{t\to\infty} \frac{\sigma\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\}}{\rho} +$$

$$+ \lim_{t\to\infty} \left[x(0) - \frac{\sigma}{\rho}\right]\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\left\{\rho\left(\frac{1-\beta}{\beta}\right)t\right\} = 0. \qquad (30)$$

Given that in the long-run x is always different from zero, the transversality condition imposes as necessary but not sufficient condition: $\lim_{t\to\infty} \theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\} = 0$. Consequently, looking at the second right-hand term of (30), we realize that the transversality condition also imposes the constraint $x(0) = \frac{\sigma}{\rho}$, from which we deduce the stationarity of x simply by substituting into (29). Moreover, this conclusion also implies a particular and well-defined initial value for the shadow price of physical capital:

$$\theta_1(0) = \left(\frac{\sigma}{\rho}\frac{1}{k_0}\right)^{\sigma}, \qquad (31)$$

where σ, the inverse of the intertemporal elasticity of substitution, is equal to the elasticity of goods production with respect to physical capital stock, β. □

Proposition 3 *Under the equilibrium conditions:*

i) if $\gamma > \beta$ and $\delta(1+\gamma-\beta) - \rho > 0$ then there exist a continuum of equilibrium paths for h starting from h_0. These paths may be characterized by the multiplicity of initial values $\theta_2(0) = (1+\epsilon)\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} h_0^{\gamma-\beta}$, where $\epsilon \gtreqless 0$ is indeterminate.

ii) if $\gamma > \beta$ and $\delta(1+\gamma-\beta) - \rho \leqslant 0$ then it does not exist any equilibrium path for h starting from h_0.

iii) if $\gamma < \beta$ and $\delta(1+\gamma-\beta) - \rho \geqslant 0$ then it does not exist any equilibrium path for h starting from h_0.

iv) if $\gamma < \beta$ *and* $\delta\,(1+\gamma-\beta)-\rho < 0$ *then there exist a unique equilibrium path for h starting from* h_0*. This unique path may be characterized by the initial value* $\theta_2\,(0) = \left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} h_0^{-(\beta-\gamma)}$.

Proof. Making use of the previous result about the instrumental variable x, we can reconsider the non-linear dynamic system (22)-(25), which may be sequentially solved in closed form. We do not need to transform this original modified Hamiltonian dynamic system by reducing its dimension. Instead, we can substitute the results from Propositions 1 and 2 in (25) getting:

$$\overset{\bullet}{h} = \delta h - \psi_1 h^{\frac{\gamma}{\beta}}, \tag{32}$$

where $\psi_1 = \left(\frac{1-\beta}{\beta}\right)\xi\theta_2^{-\frac{1}{\beta}}(0)\frac{\sigma}{\rho}\exp\left\{\frac{\delta-\rho}{\beta}t\right\}$. Equation (32) may be solved in two steps using Bernoulli's method, which leads to the general solution:

$$h = \left\{\left[h_0^{\frac{\beta-\gamma}{\beta}} + W_1\right]\exp\left\{\frac{\delta\,(\beta-\gamma)}{\beta}t\right\} - W_1\exp\left\{\frac{\delta-\rho}{\beta}t\right\}\right\}^{\frac{\beta}{\beta-\gamma}}, \tag{33}$$

where:

$$W_1 = -\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\,\xi\theta_2^{-\frac{1}{\beta}}(0)\frac{\sigma}{\rho}}{\delta\,(1+\gamma-\beta)-\rho}.$$

The transversality condition (9), in turn, may be written as:

$$0 = \lim_{t\to\infty}\,[\,(\theta_2(0)h_0)^{\frac{\beta-\gamma}{\beta}} - \frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\,\xi\theta_2^{-\frac{1+\gamma-\beta}{\beta}}(0)\frac{\sigma}{\rho}}{\delta\,(1+\gamma-\beta)-\rho} +$$

$$+\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\,\xi\theta_2^{-\frac{1+\gamma-\beta}{\beta}}(0)\frac{\sigma}{\rho}}{\delta\,(1+\gamma-\beta)-\rho}\exp\left\{\frac{\delta\,(1+\gamma-\beta)-\rho}{\beta}t\right\}]^{\frac{\beta}{\beta-\gamma}}, \tag{34}$$

and the different cases in Proposition 3 arise almost automatically in a natural way. □

As we have seen in Proposition 3, the initial value for the shadow price of human capital admits the general specification:

$$\theta_2(0) = (1+\epsilon)\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\,\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} h_0^{\gamma-\beta}. \tag{35}$$

This expression will correspond to case *i)* under the additional constraints: $\gamma > \beta$ and $\delta(1+\gamma-\beta) - \rho > 0$ for any $\epsilon \gtrless 0$. Moreover, it will correspond to case *iv)* under the alternative set of constraints: $\epsilon = 0$, $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$. On the other hand, the coefficient W_1 appearing in (33) may be simplified by defining $W_1 = -(1+\Delta)\, h_0^{\frac{\beta-\gamma}{\beta}}$, where $1+\Delta \equiv (1+\epsilon)^{-\frac{1}{\beta}}$ and $\Delta \gtrless 0$ depending on whether $\epsilon \lessgtr 0$. Now, we can use this definition to derive a general expression for h, which encompasses the two cases *i)* and *iv)* from Proposition 3:

$$h = \frac{h_0}{\left[1+\Delta - \Delta\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{\beta}{\gamma-\beta}}}\exp\left\{\frac{\rho-\delta}{\gamma-\beta}t\right\}. \tag{36}$$

This expression will correspond to case *i)* under the constraints: $\gamma > \beta$ and $\delta(1+\gamma-\beta) - \rho > 0$, for any $\Delta \gtrless 0$. It shows a multiplicity of solution trajectories for h because of the indeterminate value of the parameter Δ. Moreover, it will correspond to case *iv)* under the constraints: $\epsilon = \Delta = 0$, $\gamma < \beta$ and $\delta(1+\gamma-\beta) - \rho < 0$, showing a unique solution trajectory for h because in this case the parameter Δ takes a definite value.

Proposition 4 *Under the equilibrium conditions:*

a) if $\gamma > \beta$ and $\delta(1+\gamma-\beta) - \rho > 0$ then there exist a continuum of equilibrium paths for θ_2. These paths may be characterized by the multiplicity of initial values $\theta_2(0) = (1+\Delta)^{-\beta}\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} h_0^{\gamma-\beta}$, where $\Delta \gtrless 0$ is indeterminate.

b) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$ then there exist a unique equilibrium path for θ_2. This unique path may be characterized by the initial value $\theta_2(0) = \left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} h_0^{-(\beta-\gamma)}$.

Otherwise it does not exist any equilibrium path for θ_2.

Proof. We only need to take (26) and substitute the value of $\theta_2(0)$ just determined in (35). Then, we get:

$$\theta_2 = (1+\Delta)^{-\beta} \left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\,\xi \frac{\sigma}{\rho}}{\delta\,(1+\gamma-\beta)-\rho} \right)^{\beta} h_0^{\gamma-\beta} \exp\left\{-\left(\delta-\rho\right)t\right\}. \tag{37}$$

Multiplicity appears associated with the indeterminate value of Δ, while in case *b)*, where $\Delta = 0$, the indetermination disappears and we find a unique trajectory. □

Lemma 1 *The equilibrium paths for θ_2 and h take only positive values if and only if $\Delta > -1$.*

Proof. From (37), given the correlation among the signs of the parameter constraints as indicated in Proposition 4, we conclude that the positiveness of θ_2 depends on the constraint $\Delta > -1$ alone. From (36), the positiveness of h also depends on the constraint $\Delta > -1$, given the sign of the parameter constraints. □

Proposition 5 *If the externality associated with the human capital stock is strong enough and $\delta\,(1+\gamma-\beta)-\rho > 0$ then any of the multiple equilibrium trajectories for h starting from h_0, while describing transitional dynamics, approaches asymptotically to an undetermined positive balanced growth path where the human capital stock grows permanently at a positive constant rate, $\bar{g}_h^I = \frac{\rho-\delta}{\gamma-\beta} > 0$, if and only if $\rho > \delta$.*

Proof. A strong externality means that $\gamma > \beta$ and, according to Proposition 3, this comes together with the constraint $\delta\,(1+\gamma-\beta)-\rho > 0$. Thus, looking at (36) we find that in the long-run any of the multiple equilibrium trajectories for h evolves transitionally approaching to its associated positive balanced growth path:

$$\bar{h}_I = \frac{h_0}{[1+\Delta]^{\frac{\beta}{\gamma-\beta}}} \exp\left\{\frac{\rho-\delta}{\gamma-\beta} t\right\}, \tag{38}$$

for any $\Delta > -1$. Along these asymptotic paths, the assumed necessary and sufficient condition for positive growth becomes obvious. □

Corollary 1 *Under the parameter constraints assumed in the previous Proposition, any of the long-run equilibrium trajectories or balanced growth paths, which implies permanent and positive growth for* h*, also implies permanent and positive growth for its associated shadow price* θ_2*. Nevertheless, along any of such trajectories we find non-explosivity because the transversality condition is satisfied.*

Proposition 6 *If the externality associated with the human capital stock is not too strong and* $\delta(1+\gamma-\beta)-\rho<0$ *then, associated with the unique equilibrium trajectory for* h *starting from* h_0*, it does not exist transitional dynamics at all, and the human capital stock grows forever along such a balanced growth path at a positive constant rate,* $\bar{g}_h^{II}=\frac{\delta-\rho}{\beta-\gamma}>0$*, if and only if* $\delta>\rho$*.*

Proof. A weak externality means that $\gamma<\beta$ and, according to Proposition 3, if $\delta(1+\gamma-\beta)-\rho<0$ then the constraint $\Delta=0$ applies too. Thus, substituting the latter in (36), we get the following positive balanced growth path:

$$h=\bar{h}_{II}=h_0\exp\left\{\frac{\delta-\rho}{\beta-\gamma}t\right\}. \tag{39}$$

Consequently, the postulated necessary and sufficient condition for positive growth becomes obvious. □

Corollary 2 *Under the parameter constraints assumed in the previous Proposition, the unique equilibrium trajectory and balanced growth path which implies permanent and positive growth for* h*, also implies a continuous decreasing movement for its associated shadow price* θ_2*. Along this trajectory the transversality condition is satisfied.*

Proposition 7 *Under the equilibrium conditions:*

I) if $\gamma>\beta$ *and* $\delta(1+\gamma-\beta)-\rho>0$ *then there exist a continuum of equilibrium paths for* θ_1 *starting from* $\theta_1(0)$*. These paths may be characterized by the indeterminate value of the parameter* Δ*.*

II) if $\gamma<\beta$ *and* $\delta(1+\gamma-\beta)-\rho<0$*, hence* $\Delta=0$*, then there exist a unique equilibrium path for* θ_1 *starting from* $\theta_1(0)$*.*

Otherwise it does not exist any equilibrium path for θ_1 *starting from* $\theta_1(0)$*.*

Proof. Using (36) for h and (37) for θ_2, we can substitute in (22), getting the non-linear differential equation:

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \psi_2\theta_1^{\frac{1}{\beta}}, \tag{40}$$

where $\psi_2 = \xi \left(\frac{1}{1+\Delta} \frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho} \right)^{\beta-1} \left[1+\Delta-\Delta\exp\left\{ -\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t \right\} \right]^{\frac{-\gamma}{\gamma-\beta}}$ $h_0^{1+\gamma-\beta}\exp\left\{ \frac{\delta-\rho}{\beta-\gamma}(1+\gamma-\beta)t \right\}$. Equation (40) may be solved as before applying Bernoulli's method which leads to the solution:

$$\theta_1 = \left[\left(\frac{\rho}{\beta}k_0 \right)^{1-\beta} + C_\Delta^0 h_0^{1+\gamma-\beta} I_\Delta(t) \right]^{\frac{-\beta}{1-\beta}} \exp\{\rho t\}, \tag{41}$$

where $C_\Delta^0 = \frac{\left(\frac{1-\beta}{\beta}\right)\xi}{(1+\Delta)^{\frac{\beta(1+\gamma-\beta)}{\gamma-\beta}} \left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho} \right)^{1-\beta}}$ is an indeterminate constant which depends on the value of parameter Δ, and $I_\Delta(t)$ represents the following definite integral which also depends on the parameter Δ:

$$I_\Delta(t) = \int_0^t \frac{\exp\left\{ -\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}s \right\}}{\left[1-\frac{\Delta}{1+\Delta}\exp\left\{ -\frac{\delta(1+\gamma-\beta)-\rho}{\beta}s \right\} \right]^{\frac{\gamma}{\gamma-\beta}}} ds. \tag{42}$$

Equation (41) gives a continuum of solution trajectories for θ_1 depending on the indeterminate value of Δ as well as on the value of the remaining structural parameters. Hence, we will study this shadow price under two sets of parameter constraints. First, consider $\gamma > \beta$, $\delta(1+\gamma-\beta)-\rho > 0$, $\rho > \delta$ and $\Delta > -1$. In this case, the necessary transversality condition (30) which imposes the non-explosivity constraint $\lim_{t\to\infty} \theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\} = 0$, given (41) may be simplified to the following necessary condition: $\lim_{t\to\infty} I_\Delta(t)\exp\left\{ -\frac{\rho}{\beta}t \right\} = 0$.

Moreover, given the above parameter constraints, we can see the integrand function in (42) as a function converging in the long-run to the pure exponential function $\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}s\right\}$. Therefore, this suggest a bound to the function $I_\Delta(t)$ as in the following integral function:

$$I_b(t) = \int_0^t \exp\left\{-\frac{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\,\beta}s\right\}\,ds =$$

$$= \frac{(\gamma-\beta)\,\beta\left(1-\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}t\right\}\right)}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}.$$

Then, given the applicability of the transversality condition in the limit as t tends to infinity, we can reconsider the previous necessary condition in terms of the bounding function just introduced, which allow us to write: $\lim_{t\to\infty} I_b(t)\exp\left\{-\frac{\rho}{\beta}t\right\} = 0$. It is easy to see that, under the prevailing set of parameter constraints, this condition always holds and no more parameter constraint is needed.

Second, consider $\gamma < \beta$, $\delta\,(1+\gamma-\beta)-\rho < 0$, $\Delta = 0$ and $\delta > \rho$. In this case (41) simplifies to:

$$\theta_1 = \left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} + C_0^0 h_0^{1+\gamma-\beta} I_0(t)\right]^{\frac{-\beta}{1-\beta}} \exp\{\rho t\},$$

where $C_0^0 = \frac{\left(\frac{1-\beta}{\beta}\right)\xi}{\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{1-\beta}} > 0$ is the value of the constant C_Δ^0 when $\Delta = 0$, and $I_0(t) = \frac{(\gamma-\beta)\beta\left(1-\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}t\right\}\right)}{\delta\beta(1+\gamma-\beta)-\gamma\rho}$ represents the solution to the integral function $I_\Delta(t)$ under $\Delta = 0$. After some substitutions and rearranging terms we find the following expression for the shadow price of physical capital:

$$\theta_1 = [\left\{\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} - \frac{(\beta-\gamma)\,\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\right\}\exp\left\{-\frac{(1-\beta)\,\rho}{\beta}t\right\} +$$

$$+\frac{(\beta-\gamma)\,\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\exp\left\{\frac{(\delta-\rho)\,(1+\gamma-\beta)}{(\beta-\gamma)}t\right\}]^{\frac{-\beta}{1-\beta}}. \qquad (43)$$

Thus, given the solution for θ_1 and the prevailing set of parameter constraints, the transversality condition (30) which imposes the non-explosivity constraint $\lim_{t\to\infty} \theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\} = 0$, will be met always with no additional constraint on the parameter values. In this case, there exist a unique equilibrium path for θ_1, starting from $\theta_1(0)$. The initial value for θ_1 depends only on k_0, as shown in (31). However, subsequent values also depend on the initial human capital stock h_0. □

Finally, using the previous results for the variables θ_1, h and θ_2 we can substitute in (24) in such a way that we get:

$$\overset{\bullet}{k} = \psi_3 k - \psi_4, \tag{44}$$

where:

$$\psi_3 = \frac{\xi}{\beta}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}\theta_1^{\frac{1}{\beta}-1}h^{\frac{\gamma}{\beta}} = \frac{1}{\beta}\psi_2\theta_1^{\frac{1-\beta}{\beta}} =$$

$$= \frac{\frac{1}{\beta}\xi\left(\frac{1}{1+\Delta}\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta-1} h_0^{1+\gamma-\beta}\exp\left\{\left(\frac{\delta-\rho}{\beta-\gamma}(1+\gamma-\beta)+\frac{\rho}{\beta}-\rho\right)t\right\}}{\left[1+\Delta-\Delta\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{\gamma}{\gamma-\beta}}\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta}+C_\Delta^0 h_0^{1+\gamma-\beta}I_\Delta(t)\right]}$$

and

$$\psi_4 = \theta_1^{-\frac{1}{\sigma}} = \left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta}+C_\Delta^0 h_0^{1+\gamma-\beta}I_\Delta(t)\right]^{\frac{1}{1-\beta}}\exp\left\{-\frac{\rho}{\beta}t\right\}.$$

The general solution to (44) is:

$$k = k_0\exp\left\{\int_0^t \psi_3(s)\ ds\right\} - \int_0^t \psi_4(r)\exp\left\{\int_r^t \psi_3(z)\ dz\right\}\ dr. \tag{45}$$

This is an exact solution for k which depends only on the parameters and the initial conditions. Nevertheless, the above expression is quite complex and we would like to find an alternative way for representing the trajectory solution for physical capital stock. We can do that by using some previous results like the one established in Proposition 2.

Proposition 8 *Under the equilibrium conditions:*

I) if $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$ then there exist a continuum of equilibrium paths for k starting from k_0. These paths may be characterized by the indeterminate value of the parameter Δ.

II) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$, hence $\Delta = 0$ and then there exist a unique equilibrium path for k starting from k_0.

Otherwise it does not exist any equilibrium path for k starting from k_0.

Proof. Given definition (27) as well as the constant value for x under the assumption $\sigma = \beta$, and the general solution for θ_1 given in (41), we can write:

$$k = \frac{\beta}{\rho}\theta_1^{-\frac{1}{\beta}} = \frac{\beta}{\rho}\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} + C_\Delta^0 h_0^{1+\gamma-\beta} I_\Delta(t)\right]^{\frac{1}{1-\beta}} \exp\left\{-\frac{\rho}{\beta}t\right\}. \tag{46}$$

Equation (46) shows a continuum of solution trajectories for k depending on the indeterminate value of the parameter Δ, which corresponds to the following set of parameter constraints: $\gamma > \beta$, $\delta(1+\gamma-\beta)-\rho > 0$, $\rho > \delta$ and $\Delta > -1$. Instead, when the prevailing set of parameter constraints is: $\gamma < \beta$, $\delta(1+\gamma-\beta)-\rho < 0$, $\Delta = 0$ and $\delta > \rho$, the expression for physical capital stock simplifies to:

$$k = \frac{\beta}{\rho}[\left\{\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} - \frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right\}\exp\left\{-\frac{(1-\beta)\rho}{\beta}t\right\} +$$

$$+\frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\exp\left\{\frac{(\delta-\rho)(1+\gamma-\beta)}{(\beta-\gamma)}t\right\}]^{\frac{1}{1-\beta}}. \tag{47}$$

In this case, there exist a unique equilibrium path for k starting from k_0. Subsequent values of k also depend on the initial human capital stock h_0.

On the other hand, given the direct dependence of k with respect to θ_1, as established by the constancy of the variable x which arises from the transversality condition, the different cases considered in Proposition 7 necessarily have to reflect the corresponding ones in Proposition 8. □

Lemma 2 *In the case where $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$, if $\Delta > -1$ and $\delta\beta(1+\gamma-\beta)-\gamma\rho < 0$, the multiple equilibrium paths for θ_1 and k take only positive values.*

Proof. Looking at (41) and (46), if C_{Δ}^{0} and $I_{\Delta}(t)$ are always positive then we get always positive values for θ_1 and k. Given the signs of the parameter constraints, both C_{Δ}^{0} and $I(t)$ are always positive if $\Delta > -1$ and $\delta\beta(1+\gamma-\beta)-\gamma\rho<0$. □

Lemma 3 *In the case where $\gamma<\beta$ and $\delta(1+\gamma-\beta)-\rho<0$, and hence $\Delta=0$, if $\delta\beta(1+\gamma-\beta)-\gamma\rho>0$, the unique equilibrium paths for θ_1 and k take only positive values.*

Proof. This is a result which arises immediately from (43) and (47). □

Proposition 9 *If the externality associated with the human capital stock is strong enough and $\delta(1+\gamma-\beta)-\rho>0$ then any of the multiple equilibrium trajectories for k starting from k_0, while describing transitional dynamics, approaches asymptotically to an undetermined positive balanced growth path where the physical capital stock grows permanently at a positive constant rate, $\bar{g}_k^I=\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)>0$, if and only if $\rho>\delta$.*

Proof. A strong externality means that $\gamma>\beta$ and, according to Proposition 7, this constraint comes together with the constraint $\delta(1+\gamma-\beta)-\rho>0$. Thus, looking at (41), we find that in the long-run any of the multiple equilibrium trajectories for θ_1 evolves transitionally approaching to its associated positive balanced growth path:

$$\bar{\theta}_{1I}=\left(-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta C_{\Delta}^{0}h_0^{1+\gamma-\beta}}\right)^{\frac{\beta}{1-\beta}}\exp\left\{\frac{-\beta(1+\gamma-\beta)}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\}, \quad (48)$$

for any $\Delta>-1$ and given $\delta\beta(1+\gamma-\beta)-\gamma\rho<0$. Consequently, given definition (27) and the constant value for x under the assumption $\sigma=\beta$, in the long-run any of the multiple equilibrium trajectories for k evolves transitionally approaching to its associated positive balanced growth path:

$$\bar{k}_I=\frac{\beta}{\rho}\left(-\frac{(\gamma-\beta)\beta C_{\Delta}^{0}h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}\exp\left\{\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\}, \quad (49)$$

for any $\Delta > -1$ and given $\delta\beta(1+\gamma-\beta)-\gamma\rho < 0$. Along these asymptotic paths, the assumed necessary and sufficient condition for positive growth becomes obvious. Moreover, these trajectories show a direct dependence on h_0 as well as on the parameter Δ but, instead, they are absolutely independent of k_0. □

Proposition 10 *If the externality associated with the human capital stock is not too strong and $\delta(1+\gamma-\beta)-\rho < 0$ then the unique equilibrium trajectory for k starting from k_0, while describing transitional dynamics, approaches asymptotically to the unique positive balanced growth path where the physical capital stock grows permanently at a positive constant rate, $\bar{g}_k^{II} = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right) > 0$, if and only if $\delta > \rho$.*

Proof. A weak externality means that $\gamma < \beta$ and, according to Proposition 7, also that $\delta(1+\gamma-\beta)-\rho < 0$. Then the constraint $\Delta = 0$ applies too. Therefore, substituting these constraints in (46) we find an expression for k which is unique and in the long-run approaches to the unique positive balanced growth path:

$$\bar{k}_{II}= \frac{\beta}{\rho}\left(\frac{(\beta-\gamma)\,\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}} \exp\left\{\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}, \qquad (50)$$

given $\delta\beta(1+\gamma-\beta)-\gamma\rho > 0$. Consequently, the postulated necessary and sufficient condition for positive growth becomes obvious. Once again, we can see that the asymptotic path for k depends on h_0 but is completely independent of k_0.

In addition, looking for a global description of the solution, we can make use of definition (27) and the constant value for x under the assumption $\sigma = \beta$. Then, we can see that in the long-run the unique trajectory for θ_1 evolves transitionally approaching to the unique positive balanced growth path:

$$\bar{\theta}_{1II}= \left(\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\,\beta C_0^0 h_0^{1+\gamma-\beta}}\right)^{\frac{\beta}{1-\beta}} \exp\left\{\frac{-\beta(1+\gamma-\beta)}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}, \quad (51)$$

given $\delta\beta(1+\gamma-\beta)-\gamma\rho > 0$. □

Corollary 3 *For any of the cases considered in the two previous Propositions, the long-run equilibrium trajectories or balanced growth paths to which asymptotically moves the physical capital stock, imply permanent and positive growth for* k *and a continuous decrease for its associated shadow price* θ_1*. In any case, these two variables always move in opposite directions and the transversality condition is satisfied.*

The three next Propositions will give us the complete solution for the two control variables of the model: the per capita consumption c, and the fraction of non-leisure time devoted to goods production u.

Proposition 11 *Under the equilibrium conditions:*

I) if $\gamma > \beta$, $\delta(1+\gamma-\beta) - \rho > 0$, $\Delta > -1$, $\delta\beta(1+\gamma-\beta) - \gamma\rho < 0$ *and* $\rho > \delta$ *then there exist a continuum of equilibrium paths for* c *starting from* $c(0) = \frac{\rho}{\beta}k_0$*. Along each equilibrium path, which may be characterized by the indeterminate value of the parameter* Δ*, per capita consumption takes only positive values. Moreover, while describing transitional dynamics, every equilibrium trajectory approaches asymptotically to an undetermined positive balanced growth path along which* c *grows permanently at a positive constant rate,* $\bar{g}_c^I = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right) > 0$*.*

II) if $\gamma < \beta$, $\delta(1+\gamma-\beta) - \rho < 0$ *and therefore* $\Delta = 0$, $\delta\beta(1+\gamma-\beta) - \gamma\rho > 0$ *and* $\delta > \rho$ *then there exist a unique equilibrium path for* c *starting from* $c(0) = \frac{\rho}{\beta}k_0$*. Along this equilibrium path per capita consumption takes only positive values. Moreover, while describing transitional dynamics, it approaches asymptotically to the unique positive balanced growth path along which* c *grows permanently at a positive constant rate,* $\bar{g}_c^{II} = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right) > 0$*.*

Otherwise it does not exist any equilibrium path for c *starting from* $c(0)$*.*

Proof. Given the control function (20), definition (27) as well as Proposition 2 which assigns a constant value to x under the assumption $\sigma = \beta$, we get:

$$c = \frac{\rho}{\beta}k. \tag{52}$$

Consequently, the above statements become a natural extension from those which have been stated for the variable per capita physical capital stock along the previous Propositions. □

Proposition 12 *Under the equilibrium conditions:*

a) if $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$ then there exist a continuum of equilibrium paths for u. These paths may be characterized by the multiplicity of initial values $u(0) = (1+\Delta)\left(\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\gamma-\beta)}\right)$, where $\Delta \gtrless 0$ is indeterminate. Moreover, any of the multiple equilibrium trajectories asymptotically approaches to the same constant value which corresponds to the unique balanced growth path.

b) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$ then there exist a unique equilibrium path for u. This unique path for which there is no transitional dynamics, may be characterized by the initial value $u(0) = -\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\beta-\gamma)}$ which also represents the unique balanced growth path.

Otherwise it does not exist any equilibrium path for u.

Proof. Take the control function (21) which, given the constancy of $x \equiv \theta_1^{\frac{1}{\sigma}} k$ according to Proposition 2 and the general solutions for h and θ_2 according to equations (36) and (37), may be reduced to the following expression:

$$u = \frac{1}{1-\frac{\Delta}{1+\Delta}\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}} \frac{\delta(1+\gamma-\beta)-\rho}{\delta(\gamma-\beta)}. \quad (53)$$

Now, we can study the explicit solution for u under the two different sets of parameter constraints for which an equilibrium solution to the non-linear dynamic system (2)-(9) does exist. First, when $\gamma > \beta$ and $\delta(1+\gamma-\beta)-\rho > 0$, the previous equation gives a continuum of solution trajectories for u because of the indeterminate value of the parameter Δ. Moreover, it is easily derived from (53) that in the long-run, any of these multiple equilibrium trajectories for u evolves transitionally approaching to the same constant path:

$$\bar{u}_I = \frac{\delta(1+\gamma-\beta)-\rho}{\delta(\gamma-\beta)}. \quad (54)$$

Second, when $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$, hence $\Delta = 0$, the indetermination disappears and we find a unique and constant equilibrium trajectory:

$$u = \bar{u}_{II} = -\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\beta-\gamma)}. \quad (55)$$

In this case, the above expression means that there is no transitional dynamics for u. This variable remains always constant. □

Proposition 13 *Under the equilibrium conditions:*

a) In the case where $\gamma > \beta$ *and* $\delta(1+\gamma-\beta)-\rho > 0$ *the variable* u *satisfies the constraint* $1 > u > 0$ *if and only if* $\rho > \delta$ *and* $\frac{\rho-\delta}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$.

b) In the case where $\gamma < \beta$ *and* $\delta(1+\gamma-\beta)-\rho < 0$*, along with* $\Delta = 0$*, the variable* u *satisfies the constraint* $1 > u > 0$ *if and only if* $\delta > \rho$.

Proof. As we have seen along the proof of the previous Proposition, in case *a)* any of the multiple equilibrium trajectories for u starting from the indeterminate value:

$$u(0) = (1+\Delta)\left(\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\gamma-\beta)}\right), \tag{56}$$

approaches monotonically to $\bar{u}_I$, as given in (54). It is immediate to prove that $1 > u(0) > 0$ if and only if $\frac{\rho-\delta}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$, but also that $1 > \bar{u}_I > 0$ if and only if $\rho > \delta$.

On the other hand, in case *b)* the variable u follows a constant trajectory associated with the initial value:

$$u = \bar{u}_{II} = u(0) = -\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\beta-\gamma)}. \tag{57}$$

In this case, the constraint $1 > u > 0$ holds if and only if $\delta > \rho$. □

Finally, the two remaining Propositions give the complete characterization of the trajectory solution for variables such as relative prices and the ratio physical to human capital stocks.

Proposition 14 *Under the equilibrium conditions:*

I) if the externality associated with the human capital stock is strong enough, $\left(\frac{\gamma-\beta}{\beta}\right)\rho > \delta(1+\gamma-\beta)-\rho > 0$, $\frac{\rho-\delta}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$ *and* $\rho > \delta$ *then there exist a continuum of equilibrium paths for the relative price* $\frac{\theta_1}{\theta_2}$*. These paths, which take only positive values, may be characterized by the multiplicity of initial values* $\frac{\theta_1(0)}{\theta_2(0)} = \frac{(1+\Delta)^\beta}{h_0^{\gamma-\beta}}\left(\frac{\delta(1+\gamma-\beta)-\rho}{k_0\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi}\right)^\beta > 0$*, with the exact value of* Δ *being indeterminate. Moreover, all of them move asymptotically towards an undetermined positive balanced growth path that monotonically* *approaches to zero as t tends to infinity. This implies that along any of the multiple equilibrium trajectories, the ratio $\frac{\theta_1}{\theta_2}$ shows transitional dynamics.*

II) if the externality associated with the human capital stock is not too strong, $-\left(\frac{\beta-\gamma}{\beta}\right)\rho < \delta(1+\gamma-\beta)-\rho < 0$ and $\delta > \rho$ then the relative price $\frac{\theta_1}{\theta_2}$ follows a unique and positive equilibrium path, starting from $\frac{\theta_1(0)}{\theta_2(0)} = h_0^{\beta-\gamma}\left(\frac{\delta(1+\gamma-\beta)-\rho}{k_0\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi}\right)^{\beta} > 0$, which moves asymptotically towards the unique positive balanced growth path that monotonically approaches to zero as t tends to infinity.

In the particular case where no externality does exist, $\gamma = 0$, the relative price $\frac{\theta_1}{\theta_2}$ follows a unique and positive equilibrium path, which approaches monotonically to a positive constant value.

In this case, we find that along the unique equilibrium path, the ratio $\frac{\theta_1}{\theta_2}$ shows transitional dynamics.

Proof of part I. A strong externality means that $\gamma > \beta$. Moreover, we have $\delta(1+\gamma-\beta)-\rho > 0$ and $\delta\beta(1+\gamma-\beta)-\gamma\rho < 0$ as well as $\rho > \delta$ and $\frac{\rho-\delta}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$. Hence, expressions for θ_2 and θ_1 taken from (37) and (41) may be used to obtain:

$$\frac{\theta_1}{\theta_2} = \frac{(1+\Delta)^{\beta}}{h_0^{\gamma-\beta}\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta}}\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} + C_{\Delta}^{0}h_0^{1+\gamma-\beta}I_{\Delta}(t)\right]^{\frac{-\beta}{1-\beta}}\exp\{\delta t\}. \tag{58}$$

The results concerning multiplicity and positivity are obvious given the previous parameter constraints. Transitional dynamics may be also checked given that each equilibrium trajectory in (58) approaches asymptotically to its associated positive balanced growth path:

$$\left(\bar{\frac{\theta_1}{\theta_2}}\right)_I = \frac{(1+\Delta)^{\beta}\left(-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta C_{\Delta}^{0}}\right)^{\frac{\beta}{1-\beta}}}{h_0^{\frac{\gamma}{1-\beta}}\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta}}\exp\left\{\frac{-\gamma}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\}. \tag{59}$$

Therefore, the relative price $\frac{\theta_1}{\theta_2}$ moves towards zero as t tends to infinity because of the externality associated with the human capital stock. In this case θ_1 decreases and θ_2 increases.

Proof of part II. A weak externality means that $\gamma < \beta$, and then the constraint $\Delta = 0$ applies together with $\delta(1+\gamma-\beta)-\rho < 0$, $\delta\beta(1+\gamma-\beta)-\gamma\rho > 0$ and $\delta > \rho$. Hence, expressions for θ_2 and θ_1 taken from (37) and (43) allow us to write the relative prices as:

$$\frac{\theta_1}{\theta_2} = \frac{h_0^{\beta-\gamma}}{\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta}}[\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} - \frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right]\exp\left\{-\frac{\delta(1-\beta)}{\beta}t\right\}+ \tag{60}$$

$$+\frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\exp\left\{\frac{\gamma}{\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}]^{\frac{-\beta}{1-\beta}}.$$

Consequently, the results about uniqueness and positivity are obvious given the previous parameter constraints. Moreover, the equilibrium trajectory in (60) approaches asymptotically to the unique positive balanced growth path:

$$\left(\frac{\bar{\theta}_1}{\theta_2}\right)_{II} = \frac{\left(\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\beta C_0^0}\right)^{\frac{\beta}{1-\beta}}}{h_0^{\frac{\gamma}{1-\beta}}\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta}}\exp\left\{\frac{-\gamma}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}. \tag{61}$$

In this case both θ_1 and θ_2 decrease, but θ_1 goes down more rapidly than θ_2. The proof of the remaining statements appearing in Proposition 14 is immediate from the two previous equations. □

Proposition 15 *Under the equilibrium conditions:*

I) if the externality associated with the human capital stock is strong enough, $\left(\frac{\gamma-\beta}{\beta}\right)\rho > \delta(1+\gamma-\beta)-\rho > 0$, $\frac{\rho-\delta}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$ *and* $\rho > \delta$ *then there exist a continuum of equilibrium paths for the ratio between physical and human capital stocks, starting from* $\frac{k_0}{h_0}$*. Each of these paths may be characterized by the indeterminate value of the parameter* Δ*, and takes only positive values. Moreover, all of them approach asymptotically to an undetermined positive balanced growth path along which the ratio* $\frac{k}{h}$ *experiences exponential monotonic growth due to the presence of the externality. This* *implies that along any of the multiple equilibrium trajectories, the ratio* $\frac{k}{h}$ *shows transitional dynamics.*

II) if the externality associated with the human capital stock is not too strong, $-\left(\frac{\beta-\gamma}{\beta}\right)\rho < \delta\left(1+\gamma-\beta\right)-\rho < 0$ *and* $\delta > \rho$ *then the ratio between physical and human capital stocks follows a unique and positive equilibrium path, starting from* $\frac{k_0}{h_0}$*, which approaches asymptotically to the unique positive balanced growth path along which the ratio* $\frac{k}{h}$ *experiences exponential monotonic growth due to the presence of the externality.*

In the particular case where no externality does exist, $\gamma = 0$*, the ratio* $\frac{k}{h}$ *follows a unique and positive equilibrium path that moves monotonically towards a positive constant value which does not depend on the initial values* k_0 *and* h_0*.*

In this case, we find that along the unique equilibrium path, the ratio $\frac{k}{h}$ *shows transitional dynamics.*

Proof of part I. A strong externality means that $\gamma > \beta$. Moreover, we have $\delta\left(1+\gamma-\beta\right)-\rho > 0$ and $\delta\beta\left(1+\gamma-\beta\right)-\gamma\rho < 0$ as well as $\rho > \delta$ and $\frac{\rho-\delta}{\delta(1+\gamma-\beta)-\rho} > \Delta > -1$. Hence, expressions for k and h taken from (46) and (36) may be used to obtain:

$$\frac{k}{h} = \frac{\beta}{\rho h_0}\left[1+\Delta-\Delta\exp\left\{-\frac{\delta\left(1+\gamma-\beta\right)-\rho}{\beta}t\right\}\right]^{\frac{\beta}{\gamma-\beta}} \tag{62}$$

$$\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta}+C_{\Delta}^{0}h_0^{1+\gamma-\beta}I_{\Delta}(t)\right]^{\frac{1}{1-\beta}}\exp\left\{\frac{\delta\beta-\gamma\rho}{\left(\gamma-\beta\right)\beta}t\right\}.$$

The results concerning multiplicity and positivity are obvious given the previous parameter constraints. Transitional dynamics may be also checked given that each equilibrium trajectory in (62) approaches asymptotically to its associated positive balanced growth path:

$$\left(\frac{\bar{k}}{h}\right)_I = \frac{\beta\left[1+\Delta\right]^{\frac{\beta}{\gamma-\beta}}h_0^{\frac{\gamma}{1-\beta}}\left(-\frac{(\gamma-\beta)\beta C_{\Delta}^{0}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}}{\rho}\exp\left\{\frac{\gamma}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\}. \tag{63}$$

Therefore, the ratio $\frac{k}{h}$ eventually grows at a constant positive rate because of the externality associated with the human capital stock. Here, both k and h increase, but k increases faster than h.

Proof of part II. A weak externality means that $\gamma < \beta$, and then the constraint $\Delta = 0$ applies together with $\delta(1+\gamma-\beta)-\rho < 0$, $\delta\beta(1+\gamma-\beta)-\gamma\rho > 0$ and $\delta > \rho$. Hence, expressions for h and k taken from (47) and (39) allow us to write the following ratio:

$$\frac{k}{h} = \frac{\beta}{\rho h_0}[\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} - \frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right]\exp\left\{-\frac{(1-\beta)(\delta\beta-\gamma\rho)}{\beta(\beta-\gamma)}t\right\} + \tag{64}$$

$$+\frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\exp\left\{\gamma\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}]^{\frac{1}{1-\beta}}$$

Consequently, the results related to uniqueness and positivity are obvious given the previous parameter constraints. Moreover, the equilibrium trajectory in (64) approaches asymptotically to the unique positive balanced growth path:

$$\left(\frac{\bar{k}}{h}\right)_{II} = \frac{\beta}{\rho}h_0^{\frac{\gamma}{1-\beta}}\left(\frac{(\beta-\gamma)\beta C_0^0}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}\exp\left\{\frac{\gamma}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}. \tag{65}$$

Once again we find that although both k and h increase, k increases more rapidly than h. The proof of the remaining statements appearing in Proposition 15 is immediate from the two previous equations. □

Remark 1 *According to Caballé and Santos (1993), we define the new variables:* $\hat{h} = h\exp\left\{-\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\} = \left[1+\Delta-\Delta\exp\left\{-\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{-\beta}{\gamma-\beta}} h_0$, *and* $\hat{k} = k\exp\left\{-\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)t\right\} = \frac{\beta}{\rho}\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} + C_\Delta^0 h_0^{1+\gamma-\beta} I_\Delta(t)\right]^{\frac{1}{1-\beta}}$ $\exp\left\{-\left(\frac{\rho}{\beta}+\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\rho-\delta}{\gamma-\beta}\right)\right)t\right\}$. *Then, given that* $\delta(1+\gamma-\beta)-\rho > 0$ *when* $\gamma > \beta$ *and* $\rho > \delta$, *the long-run balanced growth path for* $\hat{h}$ *will correspond to* $\lim_{t\to\infty}\hat{h} = (1+\Delta)^{\frac{-\beta}{\gamma-\beta}} h_0$, *and the long-run balanced growth path for* $\hat{k}$ *will be*

$\lim_{t\to\infty}\hat{k}=\frac{\beta}{\rho}\left(-\frac{(\gamma-\beta)\beta C_{\Delta}^{0}h_{0}^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}$. *Consequently, the usual results associated with case I which imply a multiplicity of balanced growth paths for* $\hat{h}$ *and* $\hat{k}$, *become apparent given the indeterminacy of* Δ. *These variable transformations allow us to compute the following long-run result:*

$$\overline{\left(\frac{\hat{k}}{\hat{h}}\right)}_{I}=\lim_{t\to\infty}\left(\frac{\hat{k}}{\hat{h}}\right)=\frac{\beta}{\rho}\left[1+\Delta\right]^{\frac{\beta}{\gamma-\beta}}\left(-\frac{(\gamma-\beta)\,\beta C_{\Delta}^{0}h_{0}^{\gamma}}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}. \qquad (66)$$

This long-run value appears undetermined because of the indeterminate value of the parameter Δ.

Remark 2 *Under the particular constraints associated with case II:* $\delta>\rho$, $\beta>\gamma$, $\delta\,(1+\gamma-\beta)-\rho<0$ *and* $\delta\beta\,(1+\gamma-\beta)-\gamma\rho>0$, *we get* $\hat{h}=h_0\ \forall t$ *and* $\hat{k}^{1-\beta}=k_0^{1-\beta}\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{\beta(\beta-\gamma)}t\right\}+\left[1-\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{\beta(\beta-\gamma)}t\right\}\right]\left(\frac{\beta}{\rho}\right)^{1-\beta}\frac{(\beta-\gamma)\beta C_{0}^{0}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\hat{h}^{1+\gamma-\beta}$. *Consequently, the long-run balanced growth path for* $\hat{h}$ *is also given by* h_0, *while the long-run balanced growth path for* $\hat{k}$ *may be determined as* $\lim_{t\to\infty}\hat{k}=\frac{\beta}{\rho}\left(\frac{(\beta-\gamma)\beta C_{0}^{0}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}\hat{h}^{\frac{1+\gamma-\beta}{1-\beta}}$. *Now, we can also compute the unique transitional solution trajectory for the ratio between the two transformed capital stocks. That is:*

$$\left(\frac{\hat{k}}{\hat{h}}\right)_{II}=\frac{\beta}{\rho}\left(\frac{(\beta-\gamma)\,\beta C_{0}^{0}h_{0}^{\gamma}}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}$$

$$\left[1+\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta}\frac{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}{(\beta-\gamma)\,\beta C_{0}^{0}h_{0}^{1+\gamma-\beta}}-1\right]\exp\left\{-\frac{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}{\beta\,(\beta-\gamma)}t\right\}\right]^{\frac{1}{1-\beta}},$$

which in the long-run takes the value:

$$\overline{\left(\frac{\hat{k}}{\hat{h}}\right)}_{II}=\lim_{t\to\infty}\left(\frac{\hat{k}}{\hat{h}}\right)_{II}=\frac{\beta}{\rho}\left(\frac{(\beta-\gamma)\,\beta C_{0}^{0}h_{0}^{\gamma}}{\delta\beta\,(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}. \qquad (67)$$

This long-run value appears completely determined by h_0 given the structural parameters of the model. It depends positively on such initial condition because of the non-linear relationship between $\hat{h}$ and $\hat{k}$, as stated previously in this remark. Moreover, in the particular case where no externality does exist, $\gamma = 0$, the expression in equation (67) represents a ray passing through the origin.

This completes the analytical closed-form solution corresponding to the competitive equilibrium of the Lucas two-sector model of endogenous growth, when there is an externality associated with the human capital accumulation that affects the production of goods. Along the previous Propositions we have shown several results, all of them derived under the simplifying assumption $\sigma = \beta$. However, given our interest in theoretical properties of the transitional dynamics and the explicit trajectories for the different variables, the above assumption is not too restrictive. In fact, we can identify, as Xie did, the following points as the most important shortcomings of this procedure: first, consumption is proportional to physical capital stock; second, the initial physical capital stock does not contribute to determine any of the long-run balanced growth paths; and third, transitional dynamics corresponding to all the variables in the model are partially simplified, although they retain the main features. Obviously, in a more general framework our model will not display exactly the same kind of results, but it may be still considered as an orientative case for the study of more realistic and complex growth processes.

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