Inference Based on SVARs Identified with Sign and Zero Restrictions: Theory and Applications

by

Jonas E. Arias*
Juan F. Rubio-Ramírez**
Daniel F. Waggoner***

Documento de Trabajo 2013-24

December 2013

* Federal Reserve Board.
** Duke University, Federal Reserve Bank of Atlanta, and FEDEA.
*** Federal Reserve Bank of Atlanta.

Los Documentos de Trabajo se distribuyen gratuitamente a las Universidades e Instituciones de Investigación que lo solicitan. No obstante están disponibles en texto completo a través de Internet: http://www.fedea.es.

These Working Paper are distributed free of charge to University Department and other Research Centres. They are also available through Internet: http://www.fedea.es.

ISSN:1696-750
Inference Based on SVARs Identified with Sign and Zero Restrictions: Theory and Applications

Jonas E. Arias  
Federal Reserve Board

Juan F. Rubio-Ramírez*  
Duke University

Daniel F. Waggoner  
Federal Reserve Bank of Atlanta

October 12, 2013

Abstract

Are optimism shocks an important source of business cycle fluctuations? Are deficit-financed tax cuts better than deficit-financed spending to increase output? These questions have been previously studied using SVARs identified with sign and zero restrictions and the answers have been positive and definite in both cases. While the identification of SVARs with sign and zero restrictions is theoretically attractive because it allows the researcher to remain agnostic with respect to the responses of the key variables of interest, we show that current implementation algorithms do not respect the agnosticism of the theory. These algorithms impose additional sign restrictions on variables that are seemingly unrestricted that bias the results and produce misleading confidence intervals. We provide an alternative and efficient algorithm that does not introduce any additional sign restriction, hence preserving the agnosticism of the theory. Without the additional restrictions, it is hard to support the claim that either optimism shocks are an important source of business cycle fluctuations or deficit-financed tax cuts work best at improving output. Our algorithm is not only correct but also faster than current ones.

*Corresponding author: Juan F. Rubio-Ramírez <juan.rubio-ramirez@duke.edu>, Economics Department, Duke University, Durham, NC 27708; 1-919-660-1865. We thank Paul Beaudry, Deokwoo Nam, Andrew Mountford, and Jian Wang for sharing supplementary material with us, and for helpful comments. We also thank Grátula Bedátula for her support and help. Without her this paper would have been impossible. The views expressed here are the authors’ and not necessarily those of the Federal Reserve Bank of Atlanta or the Board of Governors of the Federal Reserve System. This paper has circulated under the title “Algorithm for Inference with Sign and Zero Restrictions.”
1 Introduction

Are optimism shocks an important source of business cycle fluctuations? Are deficit-financed tax cuts better than deficit-financed spending to increase output? Several questions such as these have been previously studied in the literature using SVARs identified by imposing sign and zero restrictions on impulse response functions and frequently the answers have been definite. For example, Beaudry et al. (2011) conclude that optimism shocks play a pivotal role in economic fluctuations and Mountford and Uhlig (2009) conclude that deficit-financed tax cuts are better for stimulating economic activity. Researchers combine SVARs with sign and zero restrictions because they allow the identification to remain agnostic with respect to the responses of key variables of interest. But this is just in theory. In practice this has not been the case.

We show that the current implementation algorithms do, in fact, introduce additional sign restrictions to the ones specified in the identification – violating the proclaimed agnosticism. The additional sign restrictions generate biased impulse response functions and artificially narrow confidence intervals around them. Hence, the researcher is going to be confident about the wrong thing. The consequence is that Beaudry et al. (2011) and Mountford and Uhlig (2009) are not agnostic as they pretend to be and that their the positive and sharp conclusions are misleading and due to these additional sign restrictions. The heart of the problem is that none of the existing algorithms correctly draws from the posterior distribution of structural parameters conditional on the sign and zero restrictions. In this paper we solve this problem by providing an algorithm that draws from the correct posterior, hence not introducing any additional sign restrictions. Absent the additional sign restrictions, it is hard to support the claim that either optimism shocks are an important source of business cycle fluctuations or deficit-financed tax cuts work best at improving output. Once you are truly agnostic Beaudry et al. (2011) and Mountford and Uhlig (2009) disappear.

In particular, we present an efficient algorithm for inference in SVARs identified with sign and zero restrictions that properly draws from the posterior distribution of structural parameters conditional on the sign and zero restrictions. We extend the sign restrictions methodology developed by Rubio-Ramírez et al. (2010) to allow for zero restrictions. As was the case in Rubio-Ramírez et al. (2010), we obtain most of our results by imposing sign and zero restrictions on the impulse response functions, but our algorithm allows for a larger class of restrictions. Two properties of the problem are relevant: (1) the set of structural parameters conditional on the sign and zero restrictions is of positive measure on the set of structural parameters conditional on the zero restrictions and (2) the posterior distribution
of structural parameters conditional on the zero restrictions is the product of the posterior distribution of the reduced-form parameters with the uniform distribution with respect to the Haar measure on the set of orthogonal matrices conditional on the zero restrictions. Drawing from the posterior of the reduced-form parameters is a well-understood problem. Our key theoretical contribution is to show how to efficiently draw from the uniform distribution with respect to the Haar measure on the set of orthogonal matrices conditional on the zero restrictions. This is the crucial step that allows us to draw from the posterior distribution of structural parameters conditional on the sign and zero restrictions and that differentiates our paper from existing algorithms.

Currently, the most widely used algorithm is Mountford and Uhlig’s (2009) penalty function approach. Instead of drawing from the posterior distribution of structural parameters conditional on the sign and zero restrictions, the penalty function approach selects a single value of the structural parameters by minimizing a loss function. We show that this approach has several drawbacks that crucially affect inference. First, the penalty function approach imposes additional sign restrictions on variables that are seemingly unrestricted — violating the proclaimed agnosticism of the identification. The additional sign restrictions bias the impulse response functions. Indeed, for a class of sign and zero restrictions we can even formally recover the additional sign restrictions. Second, because it chooses a single value of structural parameters, the penalty function approach creates artificially narrow confidence intervals around the impulse response functions that severely affect inference and the economic interpretation of the results. Baumeister and Benati (2010) and Benati (2013) provide another algorithm that, although it does not minimize any loss function, suffers from limitations similar to those of the penalty function approach because it also does not draw from the posterior distribution of structural parameters conditional on the sign and zero restrictions.

We show the capabilities of our algorithm and the problems of the penalty function approach by means of two applications previously analyzed in the literature using the penalty function approach. The first application is related to optimism shocks. The aim of Beaudry et al. (2011) is to provide new evidence on the relevance of optimism shocks as an important driver of macroeconomic fluctuations. In their most basic identification scheme, the authors claim to be agnostic about the response of consumption and hours to optimism shocks. Beaudry et al. (2011) conclude that optimism shocks are clearly important for explaining standard business cycle type phenomena because they increase consumption and hours. Unfortunately, we show that the positive and sharp responses of consumption and hours reported in Beaudry et al. (2011) are due to the additional sign restrictions on these variables introduced by the penalty function approach that bias impulse response functions and create
artificially narrow confidence intervals around them. Since these restrictions on consumption and hours were not part of the identification scheme, the penalty function approach contravenes the proclaimed agnosticism of the identification. Once you are truly agnostic using our methodology, Beaudry et al.’s (2011) conclusion is very hard to support.

The second application identifies fiscal policy shocks, as in Mountford and Uhlig (2009), in order to analyze the effects of these shocks on economic activity. Government revenue and government spending shocks are identified by imposing sign restrictions on the fiscal variables themselves as well as imposing orthogonality to a business cycle shock and a monetary policy shock. The identification pretends to remain agnostic with respect to the responses of output and other variables of interest to the fiscal policy shocks. Mountford and Uhlig’s (2009) main finding is that deficit-financed tax cuts work best among the different fiscal policies aimed at improving output. Analogously to Beaudry et al. (2011), the penalty function approach introduces additional sign restrictions on the response of output and other variables to fiscal policy shocks again conflicting with the acclaimed agnosticism of the identification strategy. As before, the results obtained in Mountford and Uhlig (2009) are due to biased impulse response functions and the artificially narrow confidence intervals around them created by the additional sign restrictions. Using our truly agnostic methodology, we show that it is very difficult to endorse Mountford and Uhlig’s (2009) results.

There is some existing literature that criticizes Mountford and Uhlig’s (2009) penalty function approach using arguments similar to the ones listed here. Binning (2013) proposes an alternative algorithm failing to provide any theoretical justification that his algorithm, in fact, draws from the posterior distribution of structural parameters conditional on the sign and zero restrictions. Caldara and Kamps (2012) also share our concerns about the penalty function approach but providing an alternative algorithm is out of the scope of their paper.

Not only does our method correctly draw from the posterior distribution of structural parameters conditional on the sign and zero restrictions but, at least for the two applications studied in this paper, it is also much faster than the penalty function approach. Our methodology is between three and ten times faster than the penalty function approach, depending on the number of sign and zero restrictions. It is also important to note that our approach can be embedded in a classical or Bayesian framework, although we follow only the latter. In addition, we wish to state that the aim of this paper is neither to dispute nor to challenge SVARs identified with sign and zero restrictions. In fact, our methodology preserves the virtues of the pure sign restriction approach developed in the work of Canova and Nicoló (2002), Uhlig (2005), and Rubio-Ramírez et al. (2010). Instead, our findings related to optimism
and fiscal policy shocks just indicate that the respective sign and zero restrictions used by Beaudry et al. (2011) and Mountford and Uhlig (2009) are not enough to accurately identify these particular structural shocks. It seems that more restrictions are needed in order to identify such shocks. Possibly zero restrictions. Finally, by characterizing the set of structural parameters conditional on sign and zero restrictions our key theoretical contribution will also be of interest to existing literature such as Faust (1998) and Barsky and Sims (2011) that identify shocks by maximizing the forecast error variance of certain variables subject to either sign or zero restrictions.

The paper is organized as follows. Section 2 shows some relevant results in the literature that we will later demonstrate to be wrong. Section 3 presents the methodology. It is here where we describe our theoretical contributions and algorithms. Section 4 offers some examples. Section 5 describes the penalty function approach and highlights its shortcomings. Section 6 presents the first of our applications. Section 7 presents the second application. Section 8 concludes.

2 Being Confident About the Wrong Thing

Beaudry et al. (2011) pretend to provide evidence on the relevance of optimism shocks as an important driver of macroeconomic fluctuations by exploiting sign and zero restrictions using the penalty function approach. More details about their work will be given in Section 6. At this point suffices to say that in their most basic model Beaudry et al. (2011) use data on total factor productivity (TFP), stock price, consumption, the real federal funds rate, and hours worked. In a first attempt, they identify optimism shocks as positively affecting the stock prices but being orthogonal to TFP at horizon zero. Hence, the identification scheme is agnostic about the response of both consumption and hours to optimism shocks. Figure 1 replicates the first block of Figure 1 in Beaudry et al. (2011). As it can be seen, both consumption and hours worked respond positively and strongly to optimism shocks. The results are also quite definite because of the narrow confidence intervals.

If right, this figure will clearly endorse those that think that optimism shocks are relevant for business cycle fluctuations. But this is not the case. In Section 6 we will show that Figure 1 is wrong. The penalty function approach introduces additional sign restrictions on consumption and hours, hence Figure 1 does not correctly reflect the impulse response functions (IRFs) associated with the agnostic identification scheme describe above. When compared with the correctly computed IRFs (as we will do in Section 6), Figure 1 reports upward biased responses of consumption and hours worked with artificially narrow confidence intervals. In that sense, Beaudry et al. (2011) are confident about the
Mountford and Uhlig (2009) analyze the effects of fiscal policy shocks using SVARs identified with sign restrictions. Using data on output, consumption, total government spending, total government revenue, real wages, investment, interest rate, adjusted reserves, prices of crude materials, and output deflator, they identify a government revenue shock as well as a government spending shock by imposing sign restrictions on the fiscal variables themselves as well as imposing orthogonality to a generic business cycle shock and a monetary policy shock. No sign restrictions are imposed on the responses of output, consumption, and investment to fiscal policy shocks. Thus, the identification remains agnostic with respect to the responses of these key variables of interest to fiscal policy shocks. Using the identified fiscal policy shocks, they report many different results that will be analyzed in Section 7. At this stage we want to focus on their comparison of fiscal policy scenarios. They compare deficit-spending shocks, where total government spending rises by 1 percent and total government revenue remains unchanged during the four quarters following the initial shock, with deficit-financed tax cut shocks, where total
government spending remains unchanged and total government revenue falls by 1 percent during the four quarters following the initial shock. More details about the fiscal policy scenarios will be provided in Section 7.

Figure 2: Mountford and Uhlig (2009) Cumulative Fiscal Multipliers

Figure 2 replicates Figure 13 in Mountford and Uhlig (2009). The figure shows that the median cumulative discounted IRF of output to a deficit-spending shock becomes negative after a few periods while it is always positive in the case of a deficit-financed tax cut shock. It also shows narrow confidence intervals. If right, this figure will strongly ratify those that think that deficit-financed tax cuts work best to improve output. But this is not the case. In Section 7 we will show that Figure 2 is, indeed, wrong. As it is the case with optimism shocks, the penalty function approach introduces additional sign restrictions on the response of output to the different fiscal policy shocks analyzed, hence Figure 2 does not correctly reflect the IRFs associated with the agnostic identification scheme described above. When compared with the correctly computed IRFs (as we will do in Section 7), Figure 2 reports biased
impulse response functions and artificially narrow confidence intervals. In that sense, Mountford and Uhlig (2009) are also confident about the wrong thing.

3 Our Methodology

This section is organized into three parts. First, we describe the model. Second, we review the efficient algorithm for inference using sign restrictions on IRFs developed in Rubio-Ramírez et al. (2010). Third, we extend this algorithm to also allow for zero restrictions. As mentioned, the algorithm proposed by Rubio-Ramírez et al. (2010) and our extension can be embedded in a classical or Bayesian framework. In this paper we follow the latter.

3.1 The Model

Consider the structural vector autoregression (SVAR) with the general form, as in Rubio-Ramírez et al. (2010)

\[ y_t' A_0 = \sum_{\ell=1}^{p} y_{t-\ell}' A_\ell + c + \varepsilon_t' \quad \text{for} \quad 1 \leq t \leq T, \] (1)

where \( y_t \) is an \( n \times 1 \) vector of endogenous variables, \( \varepsilon_t \) an \( n \times 1 \) vector of exogenous structural shocks, \( A_\ell \) an \( n \times n \) matrix of parameters for \( 0 \leq \ell \leq p \), \( c \) is a \( 1 \times n \) vector of parameters, \( p \) is the lag length, and \( T \) is the sample size. The vector \( \varepsilon_t \), conditional on past information and the initial conditions \( y_{0,...,y_{1-p}} \), is Gaussian with mean zero and covariance matrix \( I_n \), the \( n \times n \) identity matrix. The model described in equation (1) can be written as

\[ y_t' A_0 = x_t' A_+ + \varepsilon_t' \quad \text{for} \quad 1 \leq t \leq T, \] (2)

where \( A_+ = \begin{bmatrix} A_1' & \cdots & A_p' & c' \end{bmatrix} \) and \( x_t = \begin{bmatrix} y_{t-1}' & \cdots & y_{t-p}' & 1 \end{bmatrix} \) for \( 1 \leq t \leq T \). The dimension of \( A_+ \) is \( m \times n \), where \( m = np + 1 \). The reduced-form representation implied by equation (2) is

\[ y_t' = x_t' B + u_t' \quad \text{for} \quad 1 \leq t \leq T, \] (3)

where \( B = A_+ A_0^{-1} \), \( u_t' = \varepsilon_t' A_0^{-1} \), and \( E[u_t u_t'] = \Sigma = (A_0 A_0')^{-1} \). The matrices \( B \) and \( \Sigma \) are the reduced-form parameters, while \( A_0 \) and \( A_+ \) are the structural parameters; we assume that \( A_0 \) is invertible.
Most of the literature imposes restrictions on the IRFs. As we will see by the end of this section, the theorems and algorithms described in this paper allow us to consider a more general class of restrictions. In any case, we will use IRFs to motivate our theory. Thus, we now characterize them. We begin by introducing IRFs at finite horizons and then do the same at the infinite horizon. Once the IRFs are defined, we will show how to impose sign restrictions. In the finite horizon case, we have the following definition.

Definition 1. Let \((A_0, A_+\) be any value of structural parameters, the IRF of the \(i\)-th variable to the \(j\)-th structural shock at finite horizon \(h\) corresponds to the element in row \(i\) and column \(j\) of the matrix

\[
L_h(A_0, A_+) = (A_0^{-1}J^TF^hJ)' ,
\]

where

\[
F = \begin{bmatrix}
A_1A_0^{-1} & I_n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{p-1}A_0^{-1} & 0 & \cdots & I_n \\
A_pA_0^{-1} & 0 & \cdots & 0
\end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix}
I_n \\
0 \\
\vdots \\
0
\end{bmatrix} .
\]

In the infinite horizon case, we assume the \(i\)-th variable is in first differences.

Definition 2. Let \((A_0, A_+)\) be any value of structural parameters, the IRF of the \(i\)-th variable to the \(j\)-th structural shock at the infinite horizon (sometimes called long-run IRF) corresponds to the element in row \(i\) and column \(j\) of the matrix

\[
L_\infty(A_0, A_+) = \left( A'_0 - \sum_{\ell=1}^{p} A'_\ell \right)^{-1} .
\]

It is important to note that \(L_h(A_0Q, A_+Q) = L_h(A_0, A_+)Q\) for \(0 \leq h \leq \infty\) and \(Q \in O(n)\), where \(O(n)\) denotes the set of all orthogonal \(n \times n\) matrices.

### 3.2 Algorithm for Sign Restrictions

Let us assume that we want to impose sign restrictions at several horizons, both finite and infinite. It is convenient to stack the IRFs for all the relevant horizons into a single matrix of dimension \(k \times n\),
which we denote by \( f(\mathbf{A}_0, \mathbf{A}_+) \). For example, if the sign restrictions are imposed at horizon zero and infinity, then

\[
f(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} L_0(\mathbf{A}_0, \mathbf{A}_+) \\ L_\infty(\mathbf{A}_0, \mathbf{A}_+) \end{bmatrix},
\]

where \( k = 2n \) in this case.

Sign restrictions on those IRFs can be represented by matrices \( \mathbf{S}_j \) for \( 1 \leq j \leq n \), where the number of columns in \( \mathbf{S}_j \) is equal to the number of rows in \( f(\mathbf{A}_0, \mathbf{A}_+) \). Usually, \( \mathbf{S}_j \) will be a selection matrix and thus will have exactly one non-zero entry in each row, though the theory will work for arbitrary \( \mathbf{S}_j \). If the rank of \( \mathbf{S}_j \) is \( s_j \), then \( s_j \) is the number of sign restrictions on the IRFs to the \( j \)-th structural shock. The total number of sign restrictions will be \( s = \sum_{j=1}^{n} s_j \). Let \( \mathbf{e}_j \) denote the \( j \)-th column of \( \mathbf{I}_n \), where \( \mathbf{I}_n \) is the identity matrix of dimension \( n \times n \).

**Definition 3.** Let \( (\mathbf{A}_0, \mathbf{A}_+) \) be any value of structural parameters. These parameters satisfy the sign restrictions if and only if

\[
\mathbf{S}_j f(\mathbf{A}_0, \mathbf{A}_+) \mathbf{e}_j > 0,
\]

for \( 1 \leq j \leq n \).

From equation (2), it is easy to see that if \( (\mathbf{A}_0, \mathbf{A}_+) \) is any set of structural parameters and \( \mathbf{Q} \) is any element of \( O(n) \), the set of orthogonal matrices, then \( (\mathbf{A}_0, \mathbf{A}_+) \) and \( (\mathbf{A}_0 \mathbf{Q}, \mathbf{A}_+ \mathbf{Q}) \) are observationally equivalent. It is also well known, e.g., Geweke (1986), that a SVAR with sign restrictions is not identified, since for any \( (\mathbf{A}_0, \mathbf{A}_+) \) that satisfy the sign restrictions, \( (\mathbf{A}_0 \mathbf{Q}, \mathbf{A}_+ \mathbf{Q}) \) will also satisfy the sign restrictions for all orthogonal matrices \( \mathbf{Q} \) sufficiently close to the identity. Therefore, the set of structural parameters conditional on the sign restrictions will be an open set of positive measure in the set of all structural parameters. This suggests the following algorithm for sampling from the posterior of structural parameters conditional on the sign restrictions.

**Algorithm 1.**

1. Draw \( (\mathbf{A}_0, \mathbf{A}_+) \) from the unrestricted posterior.

2. Keep the draw if the sign restrictions are satisfied.

3. Return to Step 1 until the required number of posterior draws satisfying the sign restrictions has been obtained.
By unrestricted posterior we mean the posterior distribution of all structural parameters before any identification scheme is considered.

The only obstacle to implementing this algorithm is an efficient technique to accomplish the first step. For instance, one could use the Gibbs sampler described in Waggoner and Zha (2003a) to directly obtain draws from the unrestricted posterior. However, this technique produces serially correlated draws. Thus, a different approach is needed. An efficient alternative is to exploit the fact that the space of all structural parameters is equivalent to the product of the space of all reduced-form parameters and \( O(n) \). This mapping is given by \((B, \Sigma, Q) \rightarrow (A_0, A_+) = (T^{-1}Q, BT^{-1}Q)\), where \(\Sigma = T'T\) is the Cholesky decomposition of \(\Sigma\) such that \(T\) is upper triangular with positive diagonal and \(Q\) is an element of \(O(n)\). Because \((T^{-1}, BT^{-1})\) is observationally equivalent to \((T^{-1}Q, BT^{-1}Q)\), the likelihood is flat over the space of orthogonal matrices. It is also the case that the likelihood at the reduced-form parameters \((B, \Sigma)\) will be equal to the likelihood at the structural parameters \((T^{-1}, BT^{-1})\). A prior on the reduced-form parameters, together with the uniform distribution with respect to the Haar measure on \(O(n)\), induces a prior on the structural parameters via the above mapping.\(^1\) If the prior on the reduced-form parameters is conjugate, then the posterior on the reduced-form parameters will have the multivariate normal inverse Wishart distribution. There are efficient algorithms for obtaining independent draws from this distribution. So all that remains to be determined is an efficient algorithm for drawing from the uniform distribution with respect to the Haar measure on \(O(n)\). Canova and Nicoló (2002), Uhlig (2005), and Rubio-Ramírez et al. (2010) propose algorithms to draw from that set. However, Rubio-Ramírez et al.’s (2010) algorithm is the only computationally feasible one for moderately large SVAR systems (e.g., \(n > 4\)).\(^2\) Rubio-Ramírez et al.’s (2010) results are based on the following theorem.

**Theorem 1.** Let \(X\) be an \(n \times n\) random matrix with each element having an independent standard normal distribution. Let \(X = QR\) be the QR decomposition of \(X\).\(^3\) The random matrix \(Q\) has the uniform distribution with respect to the Haar measure on \(O(n)\).

Proof. The proof follows directly from Stewart (1980). \(\square\)

The previous discussion and Theorem 1 motivate us to modify the first step in Algorithm 1 applied to \((A_0, A_+) = (T^{-1}, BT^{-1})\) to obtain the following efficient algorithm.

---

\(^1\)The Haar measure is the unique measure on \(O(n)\) that is invariant under rotations and reflections such that the measure of all of \(O(n)\) is one. See Krantz and Parks (2008) for more details.

\(^2\)See Rubio-Ramírez et al. (2010) for details.

\(^3\)With probability one the random matrix \(X\) will be non-singular and so the QR decomposition will be unique if the diagonal of \(R\) is normalized to be positive.
Algorithm 2.

1. Draw \((B, \Sigma)\) from the posterior distribution of the reduced-form parameters.

2. Use Theorem 1 to draw an orthogonal matrix \(Q\).

3. Keep the draw if \(S_j f(T^{-1}Q, BT^{-1}Q)e_j > 0\) are satisfied for \(1 \leq j \leq n\).

4. Return to Step 1 until the required number of posterior draws satisfying the sign restrictions has been obtained.

Theorem 1 and Algorithm 2 are replications of Theorem 9 and Algorithm 2 in Rubio-Ramírez et al. (2010). As in Rubio-Ramírez et al. (2010), instead of working with the reduced-form parameters, one could work directly with the structural parameters as long as one is careful in interpreting the priors. Any original prior on the structural parameters will imply a prior on the reduced-form parameters via the mapping \((A_0, A_+) \rightarrow (B, \Sigma) = (A_+A_0^{-1}, (A_0A_0'^{-1})^{-1})\). This prior on the reduced-form parameters together with the uniform distribution with respect to the Haar measure on \(O(n)\) will induce a prior on the structural parameters. If we choose to work directly with the structural parameters, it is this induced prior that we will be implicitly using. This induced prior on the structural parameters will be equal to the original prior on the structural parameters if and only if the original prior has the property that the priors at \((A_0, A_+)\) and \((A_0Q, A_+Q)\) are equal for every orthogonal matrix \(Q\). Conjugate priors on the unrestricted structural parameters will have this property. Thus, in particular, any prior defined via dummy observations, such as the Sims-Zha prior, will have this property. Often such priors are combined with zero restrictions on the contemporaneous coefficients \(A_0\). These priors will not satisfy this property because \((A_0, A_+)\) could satisfy the zero restrictions, while \((A_0Q, A_+Q)\) will not for almost all orthogonal matrices \(Q\). While the examples considered in this paper impose zero restrictions only on the IRFs, as we will see below, the theory also allows us to also impose zero restrictions on \(A_0\) or \(A_+\). This allows us to work directly with the structural parameters using many of the priors on the structural parameters appearing in the literature. We can start with the conjugate prior on the unrestricted structural parameters and then impose zero restrictions on \(A_0\) using our methods. However, because the algorithms for drawing from the posterior of the structural parameters coming from such priors usually produce serially correlated draws, we prefer to work with the reduced-form parameters where we can obtain independent draws.

At this point it is useful to understand how Theorem 1 and Algorithm 2 work, and more important, how they can be implemented recursively. Let \(X = QR\) be the QR decomposition of \(X\) and let
\( \mathbf{x}_j = \mathbf{Xe}_j \) and \( \mathbf{q}_j = \mathbf{Qe}_j \) for \( 1 \leq j \leq n \). The \( \mathbf{q}_j \) can be obtained recursively using the Gram-Schmidt process, which is given by

\[
\mathbf{q}_j = \frac{(\mathbf{I}_n - \mathbf{Q}_{j-1}\mathbf{Q}'_{j-1}) \mathbf{x}_j}{\mathbf{x}_j} = \frac{\mathbf{N}_{j-1}\mathbf{N}'_{j-1}\mathbf{x}_j}{\mathbf{N}_{j-1}\mathbf{N}'_{j-1}} = \frac{\mathbf{N}_{j-1}\mathbf{N}'_{j-1}\mathbf{x}_j}{\mathbf{N}_{j-1}\mathbf{N}'_{j-1}} \quad \text{for} \quad 1 \leq j \leq n,
\]

where \(| | | \) is the Euclidean metric, \( \mathbf{Q}_{j-1} = \begin{bmatrix} \mathbf{q}_1 & \ldots & \mathbf{q}_{j-1} \end{bmatrix} \), and \( \mathbf{N}_{j-1} \) is any \( n \times (n - j + 1) \) matrix whose columns form an orthonormal basis for the null space of \( \mathbf{Q}'_{j-1} \).\(^4\) We follow the convention that \( \mathbf{Q}_0 \) is the \( n \times 0 \) empty matrix, \( \mathbf{Q}_0\mathbf{Q}'_0 \) is the \( n \times n \) zero matrix, and \( \mathbf{N}_0 \) is the \( n \times n \) identity matrix. Geometrically, \( \mathbf{q}_j \) is the projection of \( \mathbf{x}_j \) onto the null space of \( \mathbf{Q}'_{j-1} \) normalized to be of unit length. Alternatively, \( \mathbf{N}'_{j-1}\mathbf{x}_j \) is a standard normal draw from \( \mathbb{R}^{n-j+1} \) and \( \mathbf{N}'_{j-1}\mathbf{x}_j / (\mathbf{N}'_{j-1}\mathbf{x}_j) \) is a draw from the uniform distribution on the unit sphere centered at the origin in \( \mathbb{R}^{n-j+1} \), which is denoted by \( S^{n-j} \). Because the columns of \( \mathbf{N}_{j-1} \) are orthonormal, multiplication by \( \mathbf{N}_{j-1} \) is a rigid transformation of \( \mathbb{R}^{n-j+1} \) into \( \mathbb{R}^n \). From this alternative geometric representation, one can see why Algorithm 2 produces uniform draws from \( O(n) \). For \( 1 \leq j \leq n \), the vector \( \mathbf{q}_j \), conditional on \( \mathbf{Q}_{j-1} \), is a draw from the uniform distribution on \( S^{n-j} \). While it is more efficient to implement Algorithm 2 in a single step via the QR decomposition, the fact that it can be implemented recursively will be of use when there are zero restrictions.\(^5\)

### 3.3 Algorithm with Sign and Zero Restrictions

Let us now assume that we also want to impose zero restrictions at several horizons, both finite and infinite. Similar to the case of sign restrictions, we use the function \( f(\mathbf{A}_0, \mathbf{A}_+) \) to stack the IRFs at the desired horizons. The function \( f(\mathbf{A}_0, \mathbf{A}_+) \) will contain IRFs for both sign and zero restrictions. Zero restrictions can be represented by matrices \( \mathbf{Z}_j \) for \( 1 \leq j \leq n \), where the number of columns in \( \mathbf{Z}_j \) is equal to the number of rows in \( f(\mathbf{A}_0, \mathbf{A}_+) \). If the rank of \( \mathbf{Z}_j \) is \( z_j \), then \( z_j \) is the number of zero restrictions associated with the \( j \)-th structural shock. The total number of zero restrictions will be \( z = \sum_{j=1}^n z_j \).

**Definition 4.** Let \( (\mathbf{A}_0, \mathbf{A}_+) \) be any value of structural parameters. These parameters satisfy the zero

\(^4\)The formula just described to obtain \( \mathbf{q}_j \) recursively for \( 1 \leq j \leq n \) implicitly imposes the normalization that the diagonal of \( \mathbf{R} \) is positive.

\(^5\)While draws from \( O(n) \) can be obtained recursively by drawing from \( S^{n-j} \) for \( 1 \leq j \leq n \), \( O(n) \) is not topologically equivalent to a product of spheres, i.e., there does not exist a continuous bijection from \( O(n) \) to \( \prod_{j=1}^n S^{n-j} \).
restrictions if and only if

\[ Z_j f (A_0, A_+) e_j = 0 \]

for \( 1 \leq j \leq n \).

We can no longer use Algorithm 1 for sampling from the posterior of structural parameters conditional on the sign and the zero restrictions, since the set of structural parameters conditional on the zero restrictions will be of measure zero in the set of all structural parameters. As we show below, as long as there are not too many zero restrictions, we will be able to directly obtain draws of structural parameters conditional on the zero restrictions. This is important for the same reasons used to motivate Algorithm 1: the set of structural parameters conditional on the sign and the zero restrictions will be of positive measure in the set of structural parameters conditional on the zero restrictions. Thus, we will be able to use the following algorithm for sampling from the posterior of structural parameters conditional on the sign and the zero restrictions.

Algorithm 3.

1. Draw \((A_0, A_+)\) from the posterior satisfying the zero restrictions.

2. Keep the draw if the sign restrictions are satisfied.

3. Return to Step 1 until the required number of posterior draws satisfying both the sign and the zero restrictions has been obtained.

By posterior satisfying the zero restrictions we mean the posterior distribution of structural parameters conditional on the zero restrictions.

The only obstacle to implementing this procedure is an efficient technique for the first step. For the same reasons put forward above, given a prior on the reduced-form parameters, the posterior satisfying the zero restrictions is the product of the posterior distribution of the reduced-form parameters with the uniform distribution with respect to the Haar measure on \(O(n)\) conditional on the zero restrictions. As argued above, drawing from the posterior distribution of the reduced-form parameters is, in general, easy. Thus, all that remains to be determined is an efficient algorithm for drawing from the uniform distribution with respect to the Haar measure on \(O(n)\) conditional on the zero restrictions.

It is important to clarify what we mean by drawing from the uniform distribution with respect to the Haar measure on \(O(n)\) conditional on the zero restrictions. We draw \((B, \Sigma)\) from the posterior
distribution of reduced-form parameters and, conditional on \((B, \Sigma)\), a matrix \(Q\) from the uniform distribution with respect to the Haar measure on \(O(n)\) such that structural parameters \((T^{-1}Q, BT^{-1}Q)\) satisfy the zero restrictions. While the set of orthogonal matrices \(Q\) that satisfy the zero restrictions will depend on the draw of the reduced-form parameters, the size of this set will not. In particular, if the orthogonal matrix \(Q = [q_1 \cdots q_n]\) is such that \((T^{-1}Q, BT^{-1}Q)\) satisfies the zero restrictions, then \(q_j\) will be an element of \(S^{n-j-z_j} \subset \mathbb{R}^n\). While there are many unit spheres centered at the origin of dimension \(n - j - z_j\) in \(\mathbb{R}^n\), they are all of the same size. It is in this sense that we are justified in calling this procedure drawing from the uniform distribution with respect to the Haar measure on \(O(n)\) conditional on the zero restrictions.

The following theorems will allow us to obtain such an algorithm for drawing from the uniform distribution with respect to the Haar measure on \(O(n)\) conditional on the zero restrictions. As was the case with sign restrictions only, instead of working with the reduced-form parameters, one could work directly with the structural parameters, although, in general, this alternative is harder. For this reason, the theory that follows is developed in terms of any value of structural parameters.

The first step is to note that the zero restrictions on the IRFs can be converted into linear restrictions on the columns of the orthogonal matrix \(Q\). To see this, let \((A_0, A_+)\) be any value of structural parameters. Note that for any matrix \(Q\), we have

\[
Z_j f(A_0 Q, A_+ Q)e_j = Z_j f(A_0, A_+) Q e_j = Z_j f(A_0, A_+) q_j
\]

for \(1 \leq j \leq n\). Therefore, the zero restrictions associated with the \(j\)-th structural shock can be expressed as linear restrictions on the \(j\)-th column of the matrix \(Q\). Thus, the zero restrictions will hold if and only if

\[
Z_j f(A_0, A_+) q_j = 0
\]

for \(1 \leq j \leq n\). In addition to equation (4), we need the resulting matrix \(Q\) to be orthonormal. This condition imposes extra linear constraints on the columns of \(Q\).

Using these two insights the next theorem shows when and how, given any value of the structural parameters \((A_0, A_+)\), we can find an orthogonal matrix \(Q\) such that \((A_0 Q, A_+ Q)\) satisfies the zero restrictions.

**Theorem 2.** Let \((A_0, A_+)\) be any value of structural parameters. The structural parameters \((A_0 Q, A_+ Q)\),
where $Q$ is orthogonal, satisfy the zero restrictions if and only if $\| q_j \| = 1$ and

$$ R_j (A_0, A_+) q_j = 0, \quad (5) $$

for $1 \leq j \leq n$, where

$$ R_j (A_0, A_+) = \begin{bmatrix} Z_j f (A_0, A_+) & Q'_{j-1} \end{bmatrix}. $$

Furthermore, if the rank of $Z_j$ is less than or equal to $n - j$, then there will be non-zero solutions of equation (5) for all values of $Q_{j-1}$.

**Proof.** The first statement follows easily from the fact that $(A_0 Q, A_+) Q$ satisfies the zero restrictions if and only if $Z_j f (A_0, A_+) q_j = 0$ and the matrix $Q$ is orthogonal if and only if $\| q_j \| = 1$ and $Q'_{j-1} q_j = 0$. The second statement follows from the fact that the rank of $R_j (A_0, A_+)$ is less than or equal to $z_j + j - 1$. Thus, if $z_j \leq n - j$, then the rank of $R_j (A_0, A_+)$ will be strictly less than $n$ and there will be non-zero solutions of equation (5). □

Whether there will be non-zero solutions of equation (5) clearly depends on the ordering of the equations (columns) of the original system, which is arbitrary. We shall only consider zero restrictions such that the equations of the original system can be ordered so that $z_j \leq n - j$. Because, when considering zero restrictions together with sign restrictions, one usually only wants to have a small number of zero restrictions, this condition will almost always be satisfied in practice. If it is the case that the system can be ordered so that $z_j \leq n - j$, then Theorem 2 implies that for any value $(A_0, A_+)$ of the structural parameters one can always find an orthogonal matrix $Q$ such that $(A_0 Q, A_+) Q$ satisfies the zero restrictions. This implies that zero restrictions impose no constraints on the reduced-form parameters but will impose constraints on the orthogonal matrix $Q$.

The second step is to show how to use the results in Theorem 2 to obtain draws from the uniform distribution with respect to the Haar measure on $O(n)$ conditional on the zero restrictions. The next theorem does that.

**Theorem 3.** Let $1 \leq j \leq n$, and let $Z_j$ represent zero restrictions with the equations of the system given by (1) ordered so that $z_j \leq n - j$. Let $(A_0, A_+)$ be any value of the structural parameters. Let $Q$ be obtained as follows.

1. Let $j = 1$. 

15
2. Find a matrix $N_{j-1}$ whose columns form an orthonormal basis for the null space of $R_j(A_0, A_+)$.

3. Draw $x_j$ from the standard normal distribution on $\mathbb{R}^n$.

4. Let $q_j = N_{j-1} (N_{j-1}'x_j / \| N_{j-1}'x_j \|)$.\(^{6}\)

5. If $j = n$ stop; otherwise, let $j = j + 1$ and move to Step 2.

Then, the random matrix $Q$ has the uniform distribution with respect to the Haar measure on $O(n)$ such that $(A_0Q, A_+Q)$ satisfies the zero restrictions.

**Proof.** By Theorem 2, $Q$ will be orthogonal and $(A_0Q, A_+Q)$ will satisfy the zero restrictions. Let $n_j$ be the number of columns in $N_{j-1}$. For almost all $(A_0, A_+)$, $n_j = n - j - z_j + 1 \geq 1$. Thus, $q_j$, conditional on $Q_{j-1}$, $A_0$, and $A_+$, is a draw from the uniform distribution on the unit sphere centered at the origin whose dimension is $n - j - z_j$ and the distribution of $Q$ will be uniform with respect to the Haar measure on $O(n)$ such that $(A_0Q, A_+Q)$ satisfies the zero restrictions. \(\Box\)

It should be clear from Theorem 3 that for each $(A_0, A_+)$ there are many orthogonal matrices $Q$ such that $(A_0Q, A_+Q)$ satisfies the zero restrictions and that the particular orthogonal matrix $Q$ to be drawn will depend on the particular draw of $x_j$ for $1 \leq j \leq n$. The fact that Theorem 3 shows how to obtain draws from the uniform distribution with respect to the Haar measure on $O(n)$ conditional on the zero restrictions is the key theoretical contribution of this paper. This contribution allows us to modify the first step in Algorithm 3 applied to $(A_0, A_+) = (T^{-1}, BT^{-1})$ to obtain posterior draws that satisfy both the zero and sign restrictions.

**Algorithm 4.**

1. Draw $(B, \Sigma)$ from the posterior distribution of the reduced-form parameters.

2. Use Theorem 3, applied to $(A_0, A_+) = (T^{-1}, BT^{-1})$, to draw an orthogonal matrix $Q$ such that $(T^{-1}Q, BT^{-1}Q)$ satisfies the zero restrictions.

3. Keep the draw if $S_j f (T^{-1}Q, BT^{-1}Q) e_j > 0$ are satisfied for $1 \leq j \leq n$.

4. Return to Step 1 until the required number of posterior draws satisfying both the sign and zero restrictions has been obtained.

\(^{6}\)Alternatively, we could draw $y_j$ from the standard normal distribution on $\mathbb{R}^{n_j}$ and get $q_j = N_{j-1}y_j / \| y_j \|$, where $n_j$ is the number of columns in $N_{j-1}$, which is a positive number. This implementation will result in an even faster procedure.
From Algorithm 4 it is easy to see that, for each \((\mathbf{B}, \Sigma)\), there is a whole distribution of IRFs such that the restrictions hold. This observation is essential in interpreting the results in Sections 6 and 7.

### 3.4 Efficiency and Normalization

Because the \(\mathbf{q}_j\)'s that form the orthogonal matrix \(\mathbf{Q}\) in step 2 of Algorithm 4 are obtained recursively when applying Theorem 3, it is possible to check if the sign restrictions are satisfied as we are drawing them. This allows us to combine steps 2 and 3 in Algorithm 4 and have an early exit back to step 1 as soon as we have a draw of \(\mathbf{q}_j\) that does not satisfy \(S_j f (\mathbf{T}^{-1}, \mathbf{B} \mathbf{T}^{-1}) \mathbf{e}_j = S_j f (\mathbf{T}^{-1}, \mathbf{B} \mathbf{T}^{-1}) \mathbf{q}_j > 0\). In larger problems in which we can order the equations so that those shocks that impose the highest number of sign restrictions on IRFs appear first, this modification can result in greater efficiency.

In implementing this modification, it is critical that upon finding a \(\mathbf{q}_j\) that violates the sign restrictions, one exits back to step 1 and obtains a new draw of the reduced-form parameters. It is tempting to implement the algorithm by simply making draws of \(\mathbf{q}_j\) until we find one that satisfies the sign restrictions. However, this will usually lead to draws from the incorrect distribution. The easiest way to see this is to note that some draws of the reduced-form parameters may have large sets of orthogonal \(\mathbf{Q}\) that satisfy both the zero and sign restrictions, while other reduced-form parameters may have small sets of orthogonal \(\mathbf{Q}\) that satisfy both the zero and sign restrictions.\(^7\) This difference should be reflected in the posterior draws, but if one draws \(\mathbf{q}_j\) until one is accepted, this will not be true.

While it is not permissible to draw orthogonal matrices \(\mathbf{Q}\) until acceptance, it is permissible to draw a fixed number of orthogonal matrices \(\mathbf{Q}\) for each reduced-form draw and then keep all that satisfy the sign restrictions. However, because drawing from the reduced-form parameters is usually very efficient, it is often best to draw one orthogonal matrix \(\mathbf{Q}\) for each reduced-form draw. One instance in which it is always more efficient to make multiple draws of the orthogonal matrix \(\mathbf{Q}\) is in the case of normalization.

If for the \(j\)-th shock there are no sign restrictions, then any \(\mathbf{q}_j\) will trivially satisfy the sign restrictions. In this case, if \(\mathbf{q}_j\) is the draw of the \(j\)-th column of the orthogonal matrix \(\mathbf{Q}\), then both \(\mathbf{q}_j\) and \(-\mathbf{q}_j\) will satisfy the sign restrictions. If for the \(j\)-th shock there is exactly one sign restriction, then for any \(\mathbf{q}_j\) either \(\mathbf{q}_j\) or \(-\mathbf{q}_j\) will satisfy the sign restriction. In this case, if \(\mathbf{q}_j\) is the draw of the \(j\)-th

\(^7\)This is very different from zero restrictions only. For any reduced-form draw, the set of orthogonal \(\mathbf{Q}\) that satisfies only the zero restrictions lies on a unit sphere centered at the origin of dimension \(n - j - z_j\), which are all of the same size.
column of the orthogonal matrix $Q$ and $q_j$ does not satisfy the sign restriction, then $-q_j$ will. If for the $j$-th shock there is more than one sign restriction, then it may be the case that neither $q_j$ nor $-q_j$ will satisfy the sign restrictions. In this case, if $q_j$ is the draw of the $j$-th column of the orthogonal matrix $Q$ and $q_j$ does not satisfy the sign, then $-q_j$ may or may not satisfy the sign restrictions. Nevertheless, it will always improve efficiency to check both $q_j$ and $-q_j$ against the sign restrictions and keep all that satisfy the restrictions. Furthermore, the more shocks there are with zero or one sign restriction, the greater the efficiency gains.

If there are no sign restrictions on the $j$-th shock, and no additional normalization rule is added, we say that the shock is unnormalized. Unnormalized shocks will always have IRFs with distributions that are symmetric about zero. Thus, if we are interested in making inferences about an IRF, then the shock associated with such an IRF should always be normalized. A single sign restriction on a shock is a normalization rule. See Waggoner and Zha (2003b) for a discussion of normalization in SVAR models and suggestions for a generic normalization rule. Finally, it is important to remember that, while it is true that normalization rules do not change the statistical properties of the reduced-form, it is the case that different normalization rules can lead to different economic interpretations.

### 3.5 A General Class of Restrictions

It is worth noting that although we have used the function $f(A_0, A_+)$ to stack the IRFs, the theorems and algorithms in this paper work for any $f(A_0, A_+)$ that satisfies the conditions described in Rubio-Ramírez et al. (2010). Hence, our theory works for any $f(A_0, A_+)$ that is admissible, regular, and strongly regular as defined below.

**Condition 1.** The function $f(A_0, A_+)$ is admissible if and only if for any $Q \in O(n)$, $f(A_0 Q, A_+ Q) = f(A_0, A_+) Q$.

**Condition 2.** The function $f(A_0, A_+)$ is regular if and only if its domain is open and the transformation is continuously differentiable with $f'(A_0, A_+)$ of rank $kn$.

**Condition 3.** The function $f(A_0, A_+)$ is strongly regular if and only if it is regular and it is dense in the set of $k \times n$ matrices.
This highlights the fact that our theorems and algorithms allow us to consider two additional classes of restrictions (in addition to restrictions on IRFs). First, there are the commonly used linear restrictions on the structural parameters themselves ($A_0, A_+$. This class of restrictions includes the triangular identification as described by Christiano et al. (1996) and the non-triangular identification as described by Sims (1986), King et al. (1994), Gordon and Leeper (1994), Bernanke and Mihov (1998), Zha (1999), and Sims and Zha (2006). Second, there are the linear restrictions on the $Q$’s themselves. For instance, in the case of the latter restrictions, one can define $f(A_0, A_+) = I_n$. This final class will be useful in comparing our methodology with some existing methods of inference.

4 Example

In this section we present an example to illustrate how to use our theorems and algorithms. We assume some sign and zero restrictions and a draw from the posterior of the reduced-form parameters in order to show how Algorithm 2 allows us to draw a $Q$ conditional on the sign restrictions, while Algorithm 4 allows us to draw a $Q$ conditional on both the sign and the zero restrictions. Consider a four-variable SVAR with one lag. The dimension and lag length of the SVAR are totally arbitrary.

4.1 A Draw from the Posterior of the Reduced-Form Parameters

Let the following $B$ and $\Sigma$ be a particular draw from the posterior of the reduced-form parameters

\[
B = \begin{bmatrix}
0.7577 & 0.7060 & 0.8235 & 0.4387 \\
0.7431 & 0.0318 & 0.6948 & 0.3816 \\
0.3922 & 0.2769 & 0.3171 & 0.7655 \\
0.6555 & 0.0462 & 0.9502 & 0.7952
\end{bmatrix}
\quad \text{and} \quad
\Sigma = \begin{bmatrix}
0.0281 & -0.0295 & 0.0029 & 0.0029 \\
-0.0295 & 3.1850 & 0.0325 & -0.0105 \\
0.0029 & 0.0325 & 0.0067 & 0.0054 \\
0.0029 & -0.0105 & 0.0054 & 0.1471
\end{bmatrix}.
\]

Let the structural parameters be $(A_0, A_+) = (T^{-1}, BT^{-1})$, hence

\[
A_0 = \begin{bmatrix}
5.9655 & 0.5911 & -1.4851 & -0.0035 \\
0 & 0.5631 & -0.1455 & 0.0321 \\
0 & 0 & 12.9098 & -2.2906 \\
0 & 0 & 0 & 2.6509
\end{bmatrix}
\quad \text{and} \quad
A_+ = \begin{bmatrix}
4.5201 & 0.8454 & 9.4033 & -0.7034 \\
4.4330 & 0.4572 & 7.8615 & -0.5815 \\
2.3397 & 0.3878 & 3.4710 & 1.3104 \\
3.9104 & 0.4135 & 11.2867 & -0.0694
\end{bmatrix}.
\]
Assume that we want to impose restrictions on the IRFs at horizon zero, two, and infinity. Hence, we compute the respective IRFs and we stack them using function \( f(A_0, A_+) \) as follows

\[
f(A_0, A_+) = \begin{bmatrix}
L_0(A_0, A_+) \\
L_2(A_0, A_+) \\
L_\infty(A_0, A_+)
\end{bmatrix} =
\begin{bmatrix}
0.1676 & 0 & 0 & 0 \\
-0.1760 & 1.7760 & 0 & 0 \\
0.0173 & 0.0200 & 0.0775 & 0 \\
0.0173 & -0.0042 & 0.0669 & 0.3772 \\
0.1355 & 1.9867 & 0.1828 & 0.5375 \\
0.0259 & 1.3115 & 0.0828 & 0.2882 \\
0.1377 & 2.1813 & 0.2131 & 0.6144 \\
0.1069 & 2.0996 & 0.1989 & 0.6281 \\
0.1091 & -0.3783 & -0.0847 & -0.2523 \\
-0.1170 & 1.2928 & -0.0599 & -0.2201 \\
-0.0422 & -0.7342 & 0.0006 & -0.1695 \\
-0.0575 & -1.1662 & 0.0362 & 0.2577
\end{bmatrix}.
\]

### 4.2 The Restrictions

Assume that we want to impose a negative sign restriction at horizon two on the response of the third variable to the second structural shock, a positive sign restriction at horizon two on the response of the fourth variable to the second structural shock, a negative sign restriction at horizon zero on the response of the second variable to the third structural shock, a positive sign restriction at horizon zero, two, and infinity on the response of the first variable to the fourth structural shock, a zero restriction at horizon zero on the response of the first and third variables to the first structural shock, and a zero restriction at horizon infinity on the response of the fourth variable to the second structural shock. These restrictions can be enforced using the matrices \( S_j \) and \( Z_j \) for \( 1 \leq j \leq n \).
\[ S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \]
\[ S_3 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]
\[ S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \]
\[ Z_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]
\[ Z_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \]

Since there are no sign restrictions associated with the first structural shock, we do not need to specify \( S_1 \). Similarly, we do not specify \( Z_3 \) and \( Z_4 \).

### 4.3 Sign Restrictions

Let us start by discussing the sign restrictions that can be enforced using Algorithm 2. Assume that we draw

\[ X = \begin{bmatrix} 0.8110 & -1.8301 & -1.0833 & -1.7793 \\ -1.9581 & 0.5305 & -1.5108 & 1.0477 \\ 1.6940 & 0.4499 & -1.8539 & 1.0776 \\ -0.6052 & -0.2418 & -1.8677 & -0.1271 \end{bmatrix}, \]

where each element is drawn from an independent standard normal distribution. Then, the matrix \( Q \) associated with the QR decomposition is

\[ Q = \begin{bmatrix} 0.2917 & -0.8809 & -0.2226 & 0.2991 \\ -0.7044 & 0.0644 & -0.4764 & 0.5223 \\ 0.6094 & 0.4264 & -0.6430 & 0.1828 \\ -0.2177 & -0.1953 & -0.5569 & -0.7774 \end{bmatrix}. \]

Note that given \( Q \) the sign restrictions are satisfied since
\[
S_2 f (A_0, A_+) q_2 = \begin{bmatrix} 0.0100 & 0.0032 \end{bmatrix} > 0, S_3 f (A_0, A_+) q_3 = 0.8068 > 0,
\]
and \[
S_4 f (A_0, A_+) q_2 = \begin{bmatrix} 0.0501 & 0.6937 & 0.0157 \end{bmatrix} > 0.
\]

As in Example 1, there is no reason to expect the zero restrictions to be satisfied for such \( Q \). Indeed, in this case they do not hold,

\[
Z_1 f (A_0, A_+) q_1 = \begin{bmatrix} 0.0489 & 0.0382 \end{bmatrix} \neq 0, \text{ and } Z_2 f (A_0, A_+) q_4 = -0.0594 \neq 0.
\]

### 4.4 Sign and Zero Restrictions

We now illustrate how to find a \( Q \) that satisfies the sign and zero restrictions based on Algorithm 4. As in Example 1, assume that in step 1 we use our draw from the posterior of the reduced-form parameters. Then, step 2 of Algorithm 4 is as follows.

1. Let \( j = 1 \).

2. Find a matrix \( N_{j-1} \) whose columns form an orthonormal basis for the null space of \( R_j (A_0, A_+) \)

\[
N_0 = \begin{bmatrix} 0 & 0.9682 & -0.2502 & 0 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}.
\]

3. Draw \( x_j \) from the standard normal distribution on \( \mathbb{R}^n, x_1 = \begin{bmatrix} 0.4395 & -0.1190 & -0.9354 & 0.0464 \end{bmatrix} \).

4. Let \( q_j = N_{j-1} (N'_{j-1} x_j / \| N'_{j-1} x_j \|), q_1 = \begin{bmatrix} 0 & 0.9018 & -0.2330 & 0.3638 \end{bmatrix} \).

5. If \( j = n \) stop; otherwise, let \( j = j + 1 \) and move to step 2.

Thus, if we repeat these steps until \( j \) equals 5, we get the following matrices:
\[
N_1 = \begin{bmatrix}
0.4645 & -0.8818 \\
0.0762 & 0.0788 \\
0.8166 & 0.4549 \\
0.3340 & 0.0960
\end{bmatrix}, \quad
N_2 = \begin{bmatrix}
0.1720 & -0.0220 \\
0.2395 & -0.3561 \\
0.9554 & 0.0756 \\
0.0181 & 0.9311
\end{bmatrix}, \quad
\text{and } N_3 = \begin{bmatrix}
-0.0854 \\
-0.4203 \\
-0.2913 \\
0.8551
\end{bmatrix}.
\]

\[
x_2 = \begin{bmatrix}
-0.6711 \\
1.5332 \\
-0.1836 \\
0.3509
\end{bmatrix}, \quad
x_3 = \begin{bmatrix}
-0.5941 \\
0.5901 \\
-1.4499 \\
-0.2632
\end{bmatrix}, \quad\text{and } x_4 = \begin{bmatrix}
0.6713 \\
-0.4112 \\
0.7989 \\
-0.0868
\end{bmatrix}.
\]

\[
q_2 = \begin{bmatrix}
-0.9849 \\
0.0498 \\
0.1651 \\
-0.0177
\end{bmatrix}, \quad
q_3 = \begin{bmatrix}
-0.1509 \\
-0.0871 \\
-0.9130 \\
-0.3689
\end{bmatrix}, \quad\text{and } q_4 = \begin{bmatrix}
0.0854 \\
0.4203 \\
0.2913 \\
-0.8551
\end{bmatrix}.
\]

In this case, the sign restrictions also hold

\[
S_{2f}(A_0, A_+) q_2 = \begin{bmatrix}
0.0027 \\
0.0210
\end{bmatrix} > 0, S_{3f}(A_0, A_+) q_3 = 0.1281 > 0,
\]

and \( S_{4f}(A_0, A_+) q_4 = \begin{bmatrix}
0.0143 \\
0.4401 \\
0.0414
\end{bmatrix} > 0.\)

Clearly, the fact that the sign restrictions hold depends on the draw of \( x_j \) for \( 1 \leq j \leq n. \)

5 The Mountford and Uhlig Methodology

In this section, we discuss the penalty function approach with the sign and zero restrictions developed by Mountford and Uhlig (2009). First, we describe the algorithm. Second, we highlight how it selects one particular orthogonal matrix \( Q \) instead of drawing from the conditional uniform distribution derived in Subsection 3.3. We also analyze the consequences of this drawback. Third, we formally
show how selecting a particular orthogonal matrix $Q$ imposes additional sign restrictions on variables that are seemingly unrestricted.

5.1 Penalty Function Approach with Sign and Zero Restrictions

Let $(A_0, A_+)$ be any draw of the structural parameters. Consider a case where the identification of the $j$-th structural shock restricts the IRF of a set of variables indexed by $I_{j,+}$ to be positive and the IRF of a set of variables indexed by $I_{j,-}$ to be negative, where $I_{j,+}$ and $I_{j,-} \subset \{0, 1, \ldots, n\}$. Furthermore, assume that the restrictions on variable $i \in I_{j,+}$ are enforced during $H_{i,j,+}$ periods and the restrictions on variable $i \in I_{j,-}$ are enforced during $H_{i,j,-}$ periods. In addition to the sign restrictions, assume that the researcher imposes zero restrictions to identify the $j$-th structural shock. Let $Z_j$ and $f(A_0, A_+)$ denote the latter. The penalty function approach finds an orthogonal matrix $\bar{Q}^*_j = [\bar{q}^*_1 \cdots \bar{q}^*_n]$ such that IRFs come close to satisfying the sign restrictions, conditional on the zero restrictions being satisfied, according to a loss function. In particular, for $1 \leq j \leq n$, this approach solves the following optimization problem

$$\bar{q}^*_j = \arg\min_{\bar{q}_j \in S} \Psi(\bar{q}_j)$$

subject to

$$Z_j f(A_0, A_+) \bar{q}_j = 0 \text{ and } \bar{Q}_{j-1}^* \bar{q}_j = 0$$

where

$$\Psi(\bar{q}_j) = \sum_{i \in I_{j,+}} \sum_{h=0}^{H_{i,+}} g \left( -\frac{e'_i \mathbf{L}_h (A_0, A_+) \bar{q}_j}{\sigma_i} \right) + \sum_{i \in I_{j,-}} \sum_{h=0}^{H_{i,-}} g \left( \frac{e'_i \mathbf{L}_h (A_0, A_+) \bar{q}_j}{\sigma_i} \right),$$

$$g(\omega) = 100\omega \text{ if } \omega \geq 0 \text{ and } g(\omega) = \omega \text{ if } \omega \leq 0, \sigma_i \text{ is the standard error of variable } i, \bar{Q}_{j-1}^* = [\bar{q}^*_1 \cdots \bar{q}^*_{j-1}] \text{ for } 1 \leq j \leq n, \text{ and } S = S^0. \text{ We follow the convention that } \bar{Q}_0^* \text{ is the the } n \times 0 \text{ empty matrix.}$$

As before, if the prior on the reduced-form parameters is conjugate, then the posterior of the reduced-form parameters will have the multivariate normal inverse Wishart distribution. There are very efficient algorithms for obtaining independent draws from this distribution; hence, normally the

---

8See Mountford and Uhlig (2009) for details.
9To obtain $\sigma_i$, we compute the standard deviation of the OLS residuals associated with the $i$-th variable.
research will use the above algorithm where \((A_0, A_+) = (T^{-1}, BT^{-1}).\)

5.2 Choosing a Single Orthogonal Matrix \(Q\)

As mentioned above, the set of structural parameters satisfying the sign and zero restrictions is of positive measure on the set of structural parameters satisfying the zero restrictions. Conditional on a draw from the posterior of the reduced-form parameters, our Algorithm 4 uses this result to draw from the uniform distribution of orthogonal matrices conditional on the zero restrictions being satisfied. The penalty function approach abstracts from using the result. Instead, given any draw of the reduced-form parameters, \((B, \Sigma),\) the penalty function chooses an optimal orthogonal matrix \(\bar{Q}^* = [\bar{q}^*_1 \ldots \bar{q}^*_n] \in O(n)\) that solves the following system of equations

\[
Z_j f (T^{-1}, BT^{-1}) \bar{q}^*_j = 0 \text{ and }
\]

\[
\Psi (\bar{q}^*_j) = \sum_{i \in I_+} \sum_{h=0}^{H_{i,+}} g \left( -\frac{e'_i L_h (T^{-1}, BT^{-1}) \bar{q}^*_j}{\sigma_i} \right) + \sum_{i \in I_{-}} \sum_{h=0}^{H_{i,-}} g \left( \frac{e'_i L_h (T^{-1}, BT^{-1}) \bar{q}^*_j}{\sigma_i} \right),
\]

for \(1 \leq j \leq n\) where, in practice, it is also the case that \((A_0, A_+) = (T^{-1}, BT^{-1})\) and \(\Psi (\bar{q}^*_j)\) is the value of the loss function at the optimal value \(\bar{q}^*_j\). Of course, the optimal orthogonal matrix that solves the system of equations is the one that minimizes the loss function.

There are, at least, three possible issues with this approach. First, the optimal orthogonal matrix \(\bar{Q}^*\) that solves the system of equations may be such that the sign restrictions do not hold. Second, since only one orthogonal matrix is chosen, the researcher is clearly not considering all possible values of the structural parameters conditional on the sign and zero restrictions. In the applications, we will see how this issue greatly affects the confidence intervals. Third, it is easy to guess that by choosing a single orthogonal matrix to minimize a loss function, we may be introducing bias on the IRFs and other statistics of interest. Assume that the IRFs of two variables to a particular shock are correlated. Then, by choosing a particular orthogonal matrix that maximizes the response of one of the variables to the shock by minimizing the loss function, we are biasing the response of the other variable to the same shock. The penalty function approach behaves as if there were additional sign restrictions on variables that are seemingly unrestricted and, hence, violates the agnosticism of any identification.
scheme being used. In general, it is hard to formally prove such a claim because the optimal orthogonal matrix, $\bar{Q}^*$, is a function of the draw of the reduced-form parameters; hence, in most cases, we will just be able to look at the correlations between IRFs. These correlations are useful in understanding any bias that one could find, but they fall short of being a formal argument. Fortunately, there are exceptions. In the next subsection, we present a class of sign and zero restrictions where this claim can be formally proved. For this class of restrictions, we will formally show how choosing a single orthogonal matrix may impose additional restrictions on variables that are seemingly unrestricted. Nevertheless, even without a formal proof for a general class of sign and zero restrictions, this is a very serious drawback because the most attractive features of sign restrictions is that one can be agnostic about the response of some variables of interest to some structural shocks. The applications will also highlight the dramatic economic implications of this final issue.

5.3 Is the Penalty Function Approach Truly Agnostic?

We now formally show how the penalty function approach imposes additional sign restrictions on variables that are seemingly unrestricted. In this sense, the procedure is not truly agnostic and introduces bias in the IRFs and other statistics of interest. As argued above, choosing a single orthogonal matrix minimizing a loss function is likely to introduce some bias. Nevertheless, it is hard to formally prove this because the optimal orthogonal matrix depends on a given draw of reduced-form parameters. Fortunately, there is a class of sign and zero restrictions for which a formal proof is indeed possible because the optimal orthogonal matrix is independent of the draw of the reduced-form parameters.

Consider a structural vector autoregression with $n$ variables, and assume that we are interested in imposing a positive sign restriction at horizon zero on the response of the second variable to the $j$-th structural shock, and a zero restriction at horizon zero on the response of the first variable to the $j$-th structural shock.\(^{10}\) Let $(\mathbf{B}, \Sigma)$ be any draw from the posterior of the reduced-form parameters. Then, to find the optimal orthogonal matrix, $\bar{Q}^*$, we need to solve the following problem

\[
\bar{q}_j^* = \arg\min_{\bar{q}_j \in S} \Psi(\bar{q}_j)
\]

subject to

\[
e_j^T L_0 (T^{-1}, B T^{-1}) \bar{q}_j = 0
\]

\(^{10}\)The order of the restrictions is not important. It is also the case that the results in this subsection hold when we have several zero restrictions and a single sign restriction identifying a particular structural shock. We choose to present the results for a single zero restriction to simplify the argument.
where

\[ \Psi (\bar{q}_j) = g \left( -\frac{e'_i L_0 (T^{-1}, BT^{-1}) \bar{q}_j}{\sigma_2} \right). \]

Note that we are identifying only one structural shock; therefore, we do not need to impose the orthogonality constraint between the different columns of \( \bar{Q}^* \).

Equation (6) implies that the optimal \( \bar{q}^*_j \) has to be such that \( e'_1 L_0 (T^{-1}, BT^{-1}) \bar{q}^*_j = e'_1 T' \bar{q}^*_j = t_{1,1} \bar{q}_{1,j} = 0 \), where the next to last equality follows because \( T' \) is lower triangular. Thus, \( \bar{q}^*_{1,j} = 0 \). To find the remaining entries of \( \bar{q}^*_j \), it is convenient to write \( e'_2 L_0 (T^{-1}, BT^{-1}) \bar{q}_j = e'_2 T' \bar{q}_j = \sum_{s=1}^2 t_{s,2} \bar{q}_{s,j} \), where the last equality follows because \( T' \) is lower triangular. Substituting \( \bar{q}^*_{1,j} = 0 \) into \( e'_2 L_0 (T^{-1}, BT^{-1}) \bar{q}_j \) yields \( t_{2,2} \bar{q}_{2,j} \). If \( -e'_2 L_0 (T^{-1}, BT^{-1}) \bar{q}_j \geq 0 \), then \( f \left( -\frac{e'_2 L_0 (T^{-1}, BT^{-1}) \bar{q}_j}{\sigma_2} \right) = -100 \frac{t_{2,2} \bar{q}_{2,j}}{\sigma_2} \); else \( f \left( -\frac{e'_2 L_0 (T^{-1}, BT^{-1}) \bar{q}_j}{\sigma_2} \right) = -\frac{t_{2,2} \bar{q}_{2,j}}{\sigma_2} \). Since, \( \bar{q}^*_{1,j} = 0 \), and \( \bar{q}^*_j \) must be belong to \( S \), it is straightforward to verify that the criterion function is minimized at \( \bar{q}^*_j = \left[ \begin{array}{c} 0 & 1 & 0 & \cdots & 0 \end{array} \right]' \).

If the penalty function approach were truly agnostic, it would impose no additional sign restrictions on the responses of other variables of interest to the \( j \)-th structural shock. In our example, this is not the case; the penalty function approach introduces additional sign restrictions on the response of other variables to the \( j \)-th structural shock. To illustrate the problem, note that we have not introduced explicit sign restrictions on any variable except for the second. Nevertheless, the response at horizon zero of the \( i \)-th variable to the \( j \)-th structural shock for \( i > 2 \) does not depends on \( \bar{q}^*_j \) and it equals

\[ e'_i L_0 (T^{-1}, BT^{-1}) \bar{q}^*_j = t_{2,i} \text{ for all } i > 2. \]

Thus, if \( t_{2,i} > 0 \) (\( t_{2,i} < 0 \)) the penalty function approach imposes an additional sign restriction on the response of the \( i \)-th variable to the \( j \)-th structural shock at horizon zero.

Finally, it is worth noting that the result that the criterion function is minimized at

\[ \bar{q}^*_j = \left[ \begin{array}{c} 0 & 1 & 0 & \cdots & 0 \end{array} \right]' \]

implies that, for this class of sign and zero restrictions, the Mountford and Uhlig (2009) methodology can be seen as a particular case of ours. Why? Because having the \( j \)-th column of the orthogonal matrix equal to \( \left[ \begin{array}{c} 0 & 1 & 0 & \cdots & 0 \end{array} \right]' \) can always be enforced by zero restrictions on the \( j \)-th column of the orthogonal matrix. In Subsection 6.3.1 we will show how to implement those restrictions in the
6 Application to Optimism Shocks

In this section, we use our methodology to study one application related to optimism shocks previously analyzed in the literature by Beaudry et al. (2011) using the penalty function approach. The aim of Beaudry et al. (2011) is to contribute to the debate regarding the source and nature of business cycles. The authors claim to provide new evidence on the relevance of optimism shocks as the main driver of macroeconomic fluctuations using sign and zero restrictions to isolate optimism shocks. At least in their benchmark identification scheme, Beaudry et al. (2011) want to be agnostic about the response of consumption and hours worked to optimism shocks. As we show below, the problem is that, by using the penalty function approach, they are not being really agnostic about the response of these two variables.

After replicating their results, we repeat their empirical exercises using our methodology — that truly respects the agnosticism of the identification scheme — to show how their main economic conclusion substantially changes. While Beaudry et al. (2011) conclude that optimism shocks are associated with standard business cycle type phenomena because they generate a simultaneous boom in output, investment, consumption, and hours worked, we show that, using our truly agnostic methodology, it is very hard to support such a claim. Moreover, they also find that optimism shocks account for a large share of the forecast error variance (FEV) of output, investment, consumption, and hours worked at several horizons. But again, once one uses our methodology such results are also substantially weakened. We also report how our methodology is not only correct, but faster than the penalty function approach.

6.1 Data and Identification Strategy

Beaudry et al. (2011) use two data sets. In the first one, they use data on TFP, stock price, consumption, the real federal funds rate, and hours worked. In the second one, they add investment and output. In both data sets, they consider three identification strategies described in Table 1.

Identification 1 is the benchmark, where optimism shocks (sometimes called bouts of optimism) are identified as positively affecting the stock prices and as being orthogonal to TFP at horizon zero. Identification 2 adds a positive response of consumption at horizon zero as an additional restriction to
Identification 1. Finally, Identification 3 adds a positive response of the real interest rate at horizon zero to Identification 2. Appendix 9.1 gives details on the priors and the data sets. Identification 1 is agnostic about the response of consumption and hours to optimism shocks. As we will see below, the penalty function approach will not respect this agnosticism.

Next, we map these identification strategies to the function \( f (A_0, A_+ ) \) and the matrices \( S_s \) and \( Z_s \) necessary to apply our methodology. Since the sign and zero restrictions are imposed at horizon zero, we have that \( f (A_0, A_+ ) = L_0 (A_0, A_+ ) \) in both data sets. The matrices \( S_s \) and \( Z_s \) are a function of the number of variables used in the SVAR. In the smaller data set, when five variables are used, the \( S_s \) matrices are

\[
S_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad S_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

for Identifications 1, 2, and 3 respectively, while the \( Z_s \) matrix is \( Z_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \). In the larger data set, the sign and zero restrictions are defined analogously.

### 6.2 IRFs

We first show replications of the IRFs reported in Beaudry et al. (2011) using the penalty function approach. Then, we analyze how the results change once we use our methodology. Sometimes we will label our methodology the ARRW methodology. Panel (a) in Figure 3 shows the IRFs of TFP, stock price, consumption, the federal funds rate, and hours worked under Identification 1 when using the penalty function approach on the first data set. This panel replicates the first block of Figure 1 in Beaudry et al. (2011). The identified shocks generate a boom in consumption and hours worked. The
response of hours worked is hump shaped. We also report 68 percent confidence intervals. Clearly, the confidence intervals associated with the IRFs do not contain zero for, at least, 20 quarters. Thus, it is easy to conclude that optimism shocks generate standard business cycle type phenomena. Panels (b) and (c) in Figure 3 show the IRFs of TFP, stock price, consumption, the federal funds rate, and hours worked under Identifications 2 and 3. These panels replicate the second and third blocks of Figure 1 in Beaudry et al. (2011). As expected, because of the addition of the sign restrictions on the IRF of consumption, the results are stronger. Using these two identification schemes we also find a positive response of consumption and a positive hump-shaped response of hours worked to optimism shocks. Furthermore, the positive responses last longer than under Identification 1 and the confidence intervals tell us that the IRFs are significantly different from zero. The findings reported in Figure 3 are robust to extending the number of variables. Figure 18 in the appendix shows the results when we consider the larger data set.

As expected, these IRFs are highlighted by Beaudry et al. (2011). In theory, Identification 1 is agnostic about the response of consumption and hours to an optimism shock while the identified shock generates a boom in consumption and hours worked. If correct, this conclusion would strongly support the view that optimism shocks are relevant for business cycles fluctuations. But, as we will show below, these IRFs are not correct. They do not reflect the IRFs associated with the agnostic identification.
scheme 1 because the penalty function approach introduces additional sign restrictions to the ones described in Table 1.

Once we use the ARRW methodology to compute the correct IRFs, the results highlighted by Beaudry et al. (2011) basically disappear. Panel (a) in Figure 4 reports the results for the first data set using the ARRW methodology under Identification 1. There are three important differences with the results reported in Beaudry et al. (2011). First, the penalty function approach chooses a very large median response of stock prices in order to minimize the loss function. Second, the median IRFs for consumption and hours worked are closer to zero when we use the ARRW methodology. Third, the confidence intervals associated with the ARRW are much larger than the ones obtained with the penalty function approach. As a consequence, using the penalty function approach, there is an upward bias in the IRFs and artificially narrow confidence intervals.

We need to consider Identifications 2 and 3 (see Panels (b) and (c) in Figure 4), which force consumption to increase after an optimism shock, to find moderate evidence of positive IRFs of consumption and hours worked. But it is still the case that the median response of stock prices is weaker, the median IRFs of consumption and hours worked are closer to zero (i.e., the upward bias persists) and the confidence intervals are still quite wide when compared with the ones reported in Beaudry et al. (2011). As reported in Figure 19, these findings are robust to considering a larger SVAR.

Figure 4: IRFs to an Optimism Shock Using the ARRW Methodology: Five-Variable SVAR
In summary, using the ARRW methodology it is hard to claim that optimism shocks trigger a boom in consumption and hours worked unless we impose a positive response of consumption at horizon zero. Even after imposing this extra positive sign restriction, the results under the ARRW methodology are much weaker. The sharp results reported in Beaudry et al. (2011) are, as indicated above, due to upward bias in the response of consumption and hours worked and artificially narrow confidence intervals associated with the penalty function approach. Once we use the ARRW methodology to solve these two problems the results disappear. Next, we show that the discrepancy has its origin on the fact that the penalty function approach does not respect the agnosticism of the identification scheme by introducing additional sign restrictions on consumption and hours.

6.2.1 Understanding the Bias and the Artificially Narrow Confidence Intervals

We now shed some light on the upward bias and the artificially narrow confidence intervals. Let us begin with the upward bias, we will focus on the five-variable SVAR. In the appendix we show that the same conclusions are obtained using the seven-variable SVAR. Figures 5 plots the median IRFs and the 68 percent confidence intervals obtained using the ARRW methodology and compares them with the median IRFs obtained using the penalty function approach. Figure 20, in the appendix, does the same for the larger SVAR. Clearly, the median IRFs constructed using the penalty function approach are close to the 84-th percentile band constructed using the ARRW methodology. It is easy to observe that the penalty function approach selects a large response of stock prices to optimism shocks in order to minimize the loss function. By choosing a large response of stock prices, the penalty function approach also induces a positive response of consumption and hours worked because the three responses are positively correlated. For the five-variable SVAR the correlation between the IRF of stock prices to an optimism shock at horizon zero with the IRF of consumption to the same shock and horizon is 0.22. In the case of hours worked it is 0.13. The correlations are 0.26 and 0.12 in the the larger SVAR. By inducing this positive response of consumption and hours worked the penalty function approach is introducing sign restrictions on these two variables and, thus, not respecting the agnosticism of the identification scheme.

Let us now consider the artificially narrow confidence intervals. We have repeated several times that the penalty function approach selects a single orthogonal matrix instead of drawing from the conditional uniform distribution. As we mentioned when describing Algorithm 4, for each draw from the posterior distribution of the reduced-form parameters, there is a distribution of IRFs conditional
Figure 5: Comparison of IRFs to an Optimism Shock: Five-Variable SVAR

Note: Median PFA refers to the median IRF obtained using the penalty function approach.

on the sign and zero restrictions holding. By selecting a single orthogonal matrix the penalty function approach takes a single IRF from such a distribution. Figure 6 plots the 68 percent probability intervals from the distribution of IRFs such that the sign and zero restrictions hold at the OLS point estimate of the reduced-form parameters. These intervals are constructed using a single value of the structural parameters, obtained from the Cholesky decomposition and the OLS point estimate of the reduced-form parameters, and several draws of the conditional uniform distribution of orthogonal matrix Q. We have generated these draws repeating steps 2 and 3 of Algorithm 4 for the single value of the structural parameters. The probability intervals are compared with the single IRFs obtained with the penalty function approach evaluated at the same value of the reduced-form parameters (i.e., the OLS point estimate). In Figure 21 in the appendix we report the results for the seven-variables SVAR.

The dashed line shows the value of the IRFs resulting from the penalty function approach. The shadow area describes the 68 percent probability intervals obtained with our methodology. No uncertainty is considered when the penalty function approach is used. In contrast, using the ARRW methodology we can see that there is an empirically relevant distribution of IRFs conditional on the sign and zero restrictions holding. Additionally, note that for some variables — such as stock price, consumption,

\footnote{We present the results only for identification 1. Similar results are obtained for the other two identification schemes and alternative point estimates.}
and hours worked — the IRFs obtained using the penalty function approach are close to the 84 percent band. Hence, once again, we can see how the penalty function approach picks a large response of stock prices and there is an upward bias in the response of consumption and hours worked. The fact that the Mountford and Uhlig (2009) methodology does not consider the distribution of IRFs is behind the narrower confidence intervals that Beaudry et al. (2011) report.

![Figure 6: Distribution of IRFs with the ARRW methodology vs Single IRFs with the Penalty Function Approach: Five-Variable SVAR](image)

**Note:** OLS PFA refers to the IRF obtained using the penalty function approach and the OLS reduced-form estimates.

We can summarize our findings in Figure 7. Each column compares the posterior distributions of IRFs at horizon zero and the median IRFs for stock prices, consumption, and hours worked for each identification using both the penalty function approach and the ARRW methodology.\(^{12}\) The posterior distributions are approximated using a kernel smoothing function.\(^{13}\) We have focused on the five-variable SVAR. Similar results apply to the seven-variable SVAR. Column 1 displays the results for Identification 1. Comparing the penalty function approach and the ARRW methodology we reach the following conclusions. First, the posterior distribution of the IRFs of stock prices obtained using the penalty function approach is centered around the right-hand tail of the distribution obtained using the ARRW methodology. The bias is even more clear looking at the median IRFs. Second, the penalty

\(^{12}\) We report average median IRFs computed using horizons 0 to 3. The bias is larger using four periods and the results are emphasized. In any case, the bias persists even if we use only horizon 0.

\(^{13}\) We use the MATLAB ksdensity function based on Bowman and Azzalini (1997).
Figure 7: Density of IRFs at Horizon Zero and Median of IRFs from Horizons Zero to Three

The penalty function approach dramatically underestimates the variance of the posterior distribution of IRFs of stock prices. These two results were expected since the penalty function approach maximizes the response of stock prices to optimism shocks in order to minimize the loss function. Since draws from the posterior distribution of the IRFs of consumption and hours worked are positively correlated with draws of the IRF of stock prices, we also observe artificially narrow and biased posterior distributions of the IRFs for consumption and hours worked. Columns 2 and 3 show the results for Identifications 2 and 3. In both cases, we reach the same conclusion. The posterior distributions of IRFs at horizon zero for stock prices, consumption, and hours worked are artificially compressed and upwardly biased when computed using the penalty function approach.
We now present additional evidence that the penalty function approach introduces signs restrictions on variables that are seemingly unrestricted, thus not respecting the agnosticism of the identification scheme. Let’s focus on the case of five-variables SVAR. Similar results apply to the seven-variable SVAR. We begin with Identification 1. Table 2 compares the posterior probabilities that the IRFs for consumption and hours worked at horizon zero are negative for the two methodologies. The IRF of consumption is never negative when we use Mountford and Uhlig’s (2009) methodology, while it is negative approximately 40 percent of the time under the ARRW methodology. The same is basically true for hours worked. These can also be seen by comparing the mean responses reported in Table 2. Therefore, Table 2 strongly supports to the argument that the penalty function approach does not respect the theoretical agnosticism of the identification scheme by introducing additional sign restrictions on these two variables. Another important result that can be found in the table is that the standard deviation of the IRFs at horizon zero is smaller under the penalty function approach. This is, of course, related to the fact that confidence intervals are wider when using the ARRW methodology.

<table>
<thead>
<tr>
<th></th>
<th>The Penalty Function Approach</th>
<th>The ARRW Methodology</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std dev</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.1034</td>
<td>0.0260</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>0.0736</td>
<td>0.0379</td>
</tr>
</tbody>
</table>

Table 2: Posterior Probabilities of Negative IRFs at Horizon Zero: Identification 1

Tables 3 and 4 repeat the exercise for Identifications 2 and 3. As we can see, the IRF of hours worked at horizon zero under Identification 2 is never negative using the penalty function approach, while it is negative in approximately 40 percent of the draws using our methodology. Hence, the penalty function approach also introduces additional sign restrictions on hours worked in the case of Identification 2.

<table>
<thead>
<tr>
<th></th>
<th>The Penalty Function Approach</th>
<th>The ARRW Methodology</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std dev</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>0.1325</td>
<td>0.0381</td>
</tr>
</tbody>
</table>

Table 3: Posterior Probabilities of Negative IRFs at Horizon Zero: Identification 2
The Penalty Function Approach

<table>
<thead>
<tr>
<th>Hours Worked</th>
<th>The Penalty Function Approach</th>
<th>The ARRW Methodology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Std dev</td>
<td>Pr(· &lt; 0)</td>
</tr>
<tr>
<td>0.1325</td>
<td>0.0381</td>
<td>0.0010</td>
</tr>
<tr>
<td>0.0613</td>
<td>0.2833</td>
<td>0.4090</td>
</tr>
</tbody>
</table>

Table 4: Posterior Probabilities of Negative IRFs at Horizon Zero: Identification 3

6.3 FEV

The fact that the penalty function approach adds sign restrictions on variables that are seemingly unrestricted and does not respect the agnosticism of the identification scheme is also reflected on the FEV reported in Beaudry et al. (2011). Let us first analyze the SVAR with the smaller data set. We compare the contribution of optimism shocks to the FEV obtained using the ARRW methodology and the penalty function approach. For ease of exposition, in Table 5 we focus on the contributions to the FEV at horizon 40.

We first consider Identification 1. Using the ARRW methodology, the median contribution of optimism shocks to the FEV of consumption and hours worked is 17 and 18 percent, respectively. In contrast, using the penalty function approach the median contributions are 26 and 31 percent, respectively. When Identification 2 is used, the median contribution of optimism shocks to the FEV of consumption and hours worked is 28 and 27 percent using our methodology, but it is equal to 71 and 62 percent using the penalty function approach. Identification 3 yields the highest contribution of optimism shocks to the FEV of consumption and hours worked, 40 and 29 percent, respectively, when using our methodology. However, these values are moderate compared to the 76 and 49 percent that we found when using the penalty function approach. Table 5 also reports the 68 percent confidence intervals. As was the case with IRFs, the confidence intervals are much wider under the ARRW methodology. They are so wide that, in some cases, it is easy to argue that optimism shocks explain little of the FEV of most relevant variables.

The results for the seven-variable SVAR are reported in the appendix, Table 13 reports results at horizon 40 and Table 14 reports additional horizons. As expected, because of the increase in the number of variables, the contribution of optimism shocks declines relative to the case of five variables. For example, using the ARRW methodology the median contribution of optimism to the FEV of output is 12, 16, and 22 percent under Identifications 1, 2, and 3, respectively. In any case, these values are remarkably lower than the ones found using the penalty function approach: 23, 59, and 60 percent.

\(^{14}\)Table 12 in the appendix reports the contributions to the FEV at additional horizons.
### The ARRW Methodology

<table>
<thead>
<tr>
<th></th>
<th>Identification 1</th>
<th>Identification 2</th>
<th>Identification 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Adjusted TFP</strong></td>
<td>0.09</td>
<td>0.12</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>[0.03, 0.22]</td>
<td>[0.04, 0.28]</td>
<td>[0.06, 0.33]</td>
</tr>
<tr>
<td><strong>Stock Price</strong></td>
<td>0.16</td>
<td>0.26</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>[0.03, 0.47]</td>
<td>[0.07, 0.58]</td>
<td>[0.09, 0.62]</td>
</tr>
<tr>
<td><strong>Consumption</strong></td>
<td>0.17</td>
<td>0.28</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>[0.02, 0.49]</td>
<td>[0.06, 0.59]</td>
<td>[0.13, 0.66]</td>
</tr>
<tr>
<td><strong>Real Interest Rate</strong></td>
<td>0.18</td>
<td>0.20</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>[0.07, 0.39]</td>
<td>[0.08, 0.40]</td>
<td>[0.09, 0.44]</td>
</tr>
<tr>
<td><strong>Hours Worked</strong></td>
<td>0.18</td>
<td>0.27</td>
<td>0.29</td>
</tr>
<tr>
<td></td>
<td>[0.04, 0.48]</td>
<td>[0.07, 0.55]</td>
<td>[0.07, 0.57]</td>
</tr>
</tbody>
</table>

### The Penalty Function Approach

<table>
<thead>
<tr>
<th></th>
<th>Identification 1</th>
<th>Identification 2</th>
<th>Identification 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Adjusted TFP</strong></td>
<td>0.17</td>
<td>0.22</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>[0.08, 0.30]</td>
<td>[0.10, 0.37]</td>
<td>[0.14, 0.43]</td>
</tr>
<tr>
<td><strong>Stock Price</strong></td>
<td>0.72</td>
<td>0.71</td>
<td>0.57</td>
</tr>
<tr>
<td></td>
<td>[0.55, 0.85]</td>
<td>[0.57, 0.82]</td>
<td>[0.42, 0.72]</td>
</tr>
<tr>
<td><strong>Consumption</strong></td>
<td>0.26</td>
<td>0.69</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>[0.13, 0.43]</td>
<td>[0.53, 0.83]</td>
<td>[0.59, 0.87]</td>
</tr>
<tr>
<td><strong>Real Interest Rate</strong></td>
<td>0.13</td>
<td>0.13</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>[0.07, 0.22]</td>
<td>[0.07, 0.22]</td>
<td>[0.29, 0.43]</td>
</tr>
<tr>
<td><strong>Hours Worked</strong></td>
<td>0.31</td>
<td>0.62</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td>[0.21, 0.44]</td>
<td>[0.48, 0.73]</td>
<td>[0.34, 0.64]</td>
</tr>
</tbody>
</table>

Table 5: Share of FEV Attributable to Optimism Shocks at Horizon 40: Five-Variable SVAR

respectively. As before, confidence intervals are much wider when using the ARRW methodology.

Summarizing, using the ARRW methodology it is easy to conclude that optimism shocks explain a very small share of the FEV of any variable in the SVAR. This conclusion contrasts with the results obtained using the penalty function approach. As was the case with the IRFs, since the penalty function approach is not truly agnostic, it induces an upward bias in the median explained share of the FEV and artificially narrow confidence intervals. It is because of these two issues that Beaudry et al. (2011) can claim that optimism shocks explain a large share of the FEV of some relevant variables. Once these two issues are corrected by the ARRW methodology, it is not possible to support such a claim. We have reported the results only for horizon 40 but Appendix 9.3 shows that these conclusions are true at any horizon.
6.3.1 Replicating the Penalty Function Approach using the ARRW Methodology

In this subsection, we show that in the case of Identification 1 the penalty function approach in Beaudry et al. (2011) can be replicated using the ARRW methodology by considering some additional restrictions on the orthogonal matrix $Q$.

Consider Identification 1 and note that there exists a closed-form solution to the minimization problem embedded in the penalty function approach as has been shown in Subsection 5.3. Specifically, the penalty function is minimized when the first column of the orthogonal matrix equals

$$\bar{q}_1^* = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix}' .$$

Thus, we can replicate Beaudry et al. (2011) using our methodology by imposing zero constraints on the first column of the orthogonal matrix $Q$. In particular, let $f(A_0, A_+ = \begin{bmatrix} L_0(A_0, A_+) & I_n \end{bmatrix}'$, where $I_n$ allows us to put the zero constraints on the orthogonal matrix $Q$. We define the matrices $S$ and $Z$ as follows

$$S_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
and
$$Z_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

The $S$ matrix is identical to the one reported in Subsection 6.1, but the $Z$ matrix has changed to reflect the additional restrictions on the first column of the orthogonal matrix $Q$.

Panel (b) in Figure 8 plots the IRFs of TFP, stock price, consumption, the federal funds rate, and hours worked under Identification 1 using the ARRW methodology with the additional restrictions on the first column of the orthogonal matrix $Q$. The results are identical to those reported in Panel (a) of Figure 3, which are reproduced in Panel (a) of Figure 8. Thus, we have shown that (under Identification 1), the penalty function approach is a particular case of our methodology with additional restrictions on the first column of the orthogonal matrix $Q$. This also applies to the seven-variable SVAR as shown in Figure in the appendix. Since putting constrains on the orthogonal matrix $Q$ is equivalent to impose additional sign restrictions, the penalty function approach does not respect the agnosticism of the identification scheme and it is natural that it produces artificially narrow confidence intervals and biased IRFs. Unfortunately, we can not show that the penalty function approach is a particular case of our methodology for all identification schemes. Nevertheless, it should be clear that the it always

---

15\textsuperscript{In fact, this is true for any identification that fulfills the conditions stated in Subsection 5.3.}
introduces additional restrictions (though they can not always be mapped into our methodology) that create artificially narrow confidence intervals and may introduce bias.

### 6.4 Computational Time

Our methodology is faster than the penalty function approach. Table 6 reports the results for the case of optimism shocks using the five-variable SVAR. The penalty function approach is approximately ten times slower than our methodology. Similar results can be found in the case of the seven-variable SVAR. Note that the computational time in Mountford and Uhlig’s (2009) methodology is a function of the non-linear solver used to solve the minimization of the penalty function. We start the non-linear optimization from eight random starting points and then we pick the best one. Mountford and Uhlig (2009) follow a similar approach in order to avoid finding a local minimum. Of course, Identification 1 could have been solved faster with the penalty function approach using the insights of Section 5.3.
Table 6: Computational Time in Seconds: Five-variable SVAR

<table>
<thead>
<tr>
<th>Identification 1</th>
<th>The Penalty Function Approach</th>
<th>The ARRW Methodology</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identification 2</td>
<td>98.17</td>
<td>9.81</td>
<td>10.01</td>
</tr>
<tr>
<td>Identification 3</td>
<td>102.10</td>
<td>10.18</td>
<td>10.03</td>
</tr>
<tr>
<td>Identification 4</td>
<td>106.69</td>
<td>10.90</td>
<td>9.79</td>
</tr>
</tbody>
</table>

7 Fiscal Policy Shocks

Let us now focus on the second application. The aim of Mountford and Uhlig (2009) is to analyze the effects of fiscal policy using SVARs. They focus on unanticipated and anticipated fiscal policy shocks. They identify an unanticipated government revenue shock as well as an unanticipated government spending shock by imposing sign restrictions on the fiscal variables themselves as well as imposing orthogonality to a generic business cycle shock and a monetary policy shock. No sign restrictions are imposed on the responses of output, consumption, and investment to fiscal policy shocks. Thus, the identification remains agnostic with respect to the responses of these key variables of interest to fiscal policy shocks. The problem is, again, that the penalty function approach is not really agnostic about the response of these variables.

They also consider three combined shocks (which are linear combinations of the unanticipated fiscal policy shocks): deficit-spending shocks, deficit-financed tax cut shocks, and balanced-budget spending shocks. These shocks are used to compare three fiscal policy scenarios of the same name. We proceed as in the case of the optimism shocks. We first replicate the results that Mountford and Uhlig (2009) obtain using the penalty function approach. Then, we repeat their empirical work using our methodology, that is truly agnostic, to show how their main results significantly change.16 Mountford and Uhlig (2009) conclude that deficit-financed tax cut shocks work best among the fiscal policy scenarios to improve GDP. In contrast, using our methodology we find no evidence to support such a claim. More generally, we find that it is very difficult to reach any conclusion about the effects of any of the three combined shocks (and therefore about the effects of any of the three associated fiscal policy scenarios) because of very wide confidence intervals around the median IRFs and the median fiscal multipliers associated with each scenario. As a consequence, any conclusion derived from Mountford and Uhlig’s (2009) results relies on artificially narrow confidence intervals associated with the penalty function approach.

Our findings also show that it is very hard to support any of Mountford and Uhlig’s (2009) claims

---

16To keep our paper at a reasonable length, we omit the analysis of the anticipated fiscal policy shocks.
about the effects of the unanticipated fiscal policy shocks. Regarding unanticipated government revenue shocks, while Mountford and Uhlig (2009) report that GDP and consumption significantly decline in response to such shocks using the penalty function approach, we find no support for such a claim using our methodology. The median IRFs of GDP, consumption, and non-residential investment to such shocks are negative using the penalty function approach, but positive using our methodology. Furthermore, wide confidence intervals invalidate any conclusion. In the case of unanticipated government spending shocks, except for investment, the median IRFs from both methodologies are quite similar to each other, but wide confidence intervals make very hard to reach any conclusion. As it was the case with Beaudry et al. (2011), we will argue that the the problem behind the bias and the artificially narrow confidence intervals is that, by using the penalty function approach, the authors are not being truly agnostic – additional sign restrictions on output, consumption, and investment are being imposed. Finally, for this application our methodology is also faster than the penalty function approach.

7.1 Data and Identification Strategy

We use the same data set as Mountford and Uhlig (2009) in order to shed light on the implications of our methodology. The data set contains 10 U.S. variables at a quarterly frequency from 1955 to 2000: GDP, private consumption, total government spending, total government revenue, real wages, private non-residential investment, interest rate, adjusted reserves, producer price index of crude materials (PPIC), and GDP deflator. The identification strategy is described in Table 7. Appendix 9.2 gives details about the estimation procedure and the data set.

<table>
<thead>
<tr>
<th>Shocks</th>
<th>Business Cycle</th>
<th>Monetary Policy</th>
<th>Gov Revenue</th>
<th>Gov Spending</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDP</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Gov Spending</td>
<td></td>
<td></td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>Total Gov Revenue</td>
<td>+</td>
<td></td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>Interest Rate</td>
<td></td>
<td>+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adjusted Reserves</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PPIC</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP Deflator</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Private Consumption</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Private Non-Res Investment</td>
<td></td>
<td>+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Real Wages</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7: Mountford and Uhlig (2009)
7.1.1 Unanticipated Fiscal Policy Shocks

We begin by describing the identification of the unanticipated fiscal policy shocks. Following Mountford and Uhlig (2009), we identify these shocks in three steps. In the first step, we identify a business cycle shock imposing four positive sign restrictions on GDP, private consumption, private non-residential investment, and total government revenue during four quarters — quarters zero to three — following the initial shock. In the second step, we identify a monetary policy shock imposing positive sign restrictions on interest rates, and negative sign restrictions on adjusted reserves, GDP deflator, and PPIC during four quarters following the initial shock. In addition, the monetary policy shock is required to be orthogonal to the business cycle shock. In the third step, we identify the unanticipated fiscal shocks. The unanticipated government revenue shock is identified imposing positive sign restrictions on the response of total government revenue during four quarters following the initial shock and requiring that the shock be orthogonal to the business cycle shock and the monetary policy shock. The unanticipated government spending shock is identified likewise. Hence, the identification is agnostic with respect to the responses of GDP, private consumption, and private non-residential investment to fiscal policy shocks. The penalty function approach will not respect this agnosticism. Importantly, the unanticipated fiscal shocks are not required to be orthogonal between them.

As in the case of optimism shocks, it is instructive to map the identification strategy to our methodology. The function \( f(A_0, A_+) \) and the matrices \( S_j \) necessary to apply our methodology are

\[
\begin{align*}
    f(A_0, A_+) &= \begin{bmatrix}
        L_0(A_0, A_+) \\
        L_1(A_0, A_+) \\
        L_2(A_0, A_+) \\
        L_3(A_0, A_+)
    \end{bmatrix} \\
    S_j &= \begin{bmatrix}
        S_{j0} & 0_{m(j),n} & 0_{m(j),n} & 0_{m(j),n} \\
        0_{m(j),n} & S_{j1} & 0_{m(j),n} & 0_{m(j),n} \\
        0_{m(j),n} & 0_{m(j),n} & S_{j2} & 0_{m(j),n} \\
        0_{m(j),n} & 0_{m(j),n} & 0_{m(j),n} & S_{j3}
    \end{bmatrix}
\end{align*}
\]

for \( j = 1, \ldots, 4 \).

where \( 0_{m(j),n} \) is an \( m(j) \) times \( n \) matrix of zeros and \( m(j) = 4 \) if \( j = 1 \) or 2 and \( m(j) = 1 \) otherwise, and
\( S_{1t} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \ S_{2t} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \ S_{3t} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{and} \ S_{4t} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \) for \( t = 0, \ldots, 3 \). In addition, we need to impose the orthogonality conditions between the shocks as described above. This is straightforward from the discussion in Section 3.2. The only challenge is that Mountford and Uhlig (2009) do not require orthogonality between the unanticipated fiscal shocks; thus, we need to accommodate our methodology to study this case. We accomplish this requiring that the unanticipated government revenue (spending) shock, associated with the third (fourth) column of \( Q \), be orthogonal to the first and second columns of \( Q \), associated with the business cycle and monetary policy shocks, respectively, without restricting other columns of \( Q \). For example, in the case of the unanticipated government spending shock, this is accomplished by modifying \( R_4 (A_0, A_+ ) \) in Theorem 2 to be equal to

\[ R_4 (A_0, A_+ ) = Q'_2. \]

The reader should note that a direct application of Theorem 2 would make \( R_4 (A_0, A_+ ) \) depend on \( Q'_3 \) instead of \( Q'_2 \). There are no zero restrictions; hence, we do not need to define any \( Z \) matrix.

Next, we describe the three linear combinations of the unanticipated fiscal policy shocks that are used to study the three fiscal policy scenarios.

### 7.1.2 Fiscal Policy Scenarios

The deficit-spending shocks (used to study the deficit-spending scenario) are a sequence of unanticipated fiscal policy shocks where total government spending rises by 1 percent and total government revenue remains unchanged during the four quarters following the initial shock. The deficit-financed tax cut shocks (used to study the deficit-financed tax cuts scenario) are a sequence of unanticipated fiscal policy shocks where total government spending remains unchanged and total government revenue falls by 1 percent during the four quarters following the initial shock. The balanced-budget spending
shocks (used to study the balanced-budget spending scenario) are a sequence of unanticipated fiscal policy shocks where total government spending rises by 1 percent and total government revenue rises by 1.28 percent during the four quarters following the initial shock.\textsuperscript{17} Let \((a_{s,t}, b_{s,t})\) for \(t = 0, \ldots, 3\) and \(s \in \{DS, DTC, BB\}\) denote the weights to be used in the linear combination of the unanticipated fiscal policy shocks to get the deficit-spending shocks (DS), the deficit-financed tax cut shocks (DTC), and the balanced-budget spending shocks (BB), respectively. For example, let us consider the case of a deficit-spending shocks. We can solve for such weights by solving the following linear system of equations

\[0.01 = \sum_{t=0}^{\tau} (e'_{GS}L_{\tau-t}(A_0, A_+)q_4a_{DS,t} + e'_{GR}L_0(A_0, A_+)q_3b_{DS,t}) \quad \text{for } \tau = 0, \ldots, 3\]

\[0 = \sum_{t=0}^{\tau} (e'_{GR}L_{\tau-t}(A_0, A_+)q_4a_{DS,t} + e'_{GR}L_0(A_0, A_+)q_3b_{DS,t}) \quad \text{for } \tau = 0, \ldots, 3\]

where \(e_{GS}\) (\(e_{GR}\)) is a unit vector with a one at the entry associated with total government spending (government revenue) in the SVAR and zeros otherwise. Then, we can use the weights \((a_{DS,t}, b_{DS,t})\) for \(t = 0, \ldots, 3\) to build the column vector associated with the deficit-spending shocks as \(q_{DS} = q_4a_{DS,t} + q_3b_{DS,t}\) for \(t = 0, \ldots, 3\). In a similar fashion, we can construct weights for the other two combined shocks and obtain the column vectors \(q_{DTC}\) and \(q_{BB}\).

\subsection*{7.2 IRFs to Unanticipated Fiscal Policy Shocks}

Let us begin examining the IRFs. We first show the replications of the IRFs reported by Mountford and Uhlig (2009) using the penalty function approach and then we analyze how the results change once we use our methodology. To save space, we do not report results on either business cycle or monetary policy shocks. Also, we refer to private consumption and to private non-residential investment as consumption and investment, respectively. Finally, and also because of space considerations, we just concentrate on the responses of GDP, consumption, and investment.

Figure 9 plots the IRFs to an unanticipated government revenue shock. Panel (a) replicates the results reported in Figure 4 in Mountford and Uhlig (2009). This panel shows that using the penalty function approach, the median IRFs of GDP, consumption, and investment are negative. Furthermore, the percentage increase in total government revenue is higher than the percentage increase in total government spending so that total government revenues and total government spending increase by the same amount during the four quarters following the initial shock.
the 68 percent confidence intervals are narrow and do not contain zero. Therefore, one can easily conclude that unanticipated government revenue shocks cause a decline in economic activity. In contrast, once we use our methodology (see panel, b), the sign of the median IRFs changes and the 68 percent confidence intervals are much wider. Thus, as in the case of optimism shocks, the penalty function approach creates bias and artificially narrow confidence intervals. In addition, the penalty function approach chooses a very large median response of total government revenue.

Figure 10 plots the IRFs to an unanticipated government spending shock. Panel (a) replicates Figure 7 in Mountford and Uhlig (2009) and it shows the median IRFs of GDP, consumption, and investment. Using the penalty approach, the median response of GDP changes from positive to negative in period 10, the response of consumption changes from zero to negative around period 12, and the response of investment is always negative. Although less than in the case of the unanticipated government revenue shocks, the median IRFs change when we use the ARRW methodology (see panel, b). The changes are important for investment (whose response is positive for 5 periods). Nevertheless, the confidence intervals are wider under our methodology and they contain zero. Analogously to the
case of unanticipated government revenue shocks, the penalty function approach introduces downward bias (at least for several quarters) in the response of investment to unanticipated government spending shocks and creates artificially narrow confidence intervals. It is also the case that the penalty function approach picks a large response of total government spending to minimize the loss function.

Summarizing, using the ARRW methodology we observe important changes in the median IRFs to unanticipated fiscal policy shocks with respect to the results reported in Mountford and Uhlig (2009). In what follows we show that the lack of agnosticism of the penalty function approach introduces a bias in the response of some variables and delivers confidence intervals that are artificially narrow. The strong results reported in Mountford and Uhlig (2009) are because of the bias and the artificially narrow confidence intervals. The truly agnostic ARRW methodology amends these two problems and the results disappear.
7.2.1 Understanding the Bias and the Artificially Narrow Confidence Intervals

As before, we now shed some light on the biases and artificially narrow confidence intervals. Let us begin with the biases related to unanticipated government revenue shock. The penalty function approach selects a large response of total government revenue to this shock in order to minimize the loss function. By selecting a large response of total government revenue, the penalty function approach is implicitly forcing a negative response of GDP, consumption, and investment because their IRFs are negatively correlated with the IRF of total government revenue. The correlations of the IRF of total government revenue to unanticipated government revenue shocks at horizon zero with the IRFs of GDP, consumption, and investment to the same shock and horizon are \(-0.12\), \(-0.11\), and \(-0.01\), respectively.

In the case of unanticipated government spending shocks the penalty function approach also selects a large response of total government spending in order to minimize the loss function. By choosing a large response, the penalty function approach is implicitly forcing a negative response of investment because its IRF is negatively correlated with the response of total government spending. The correlation between the IRF of total government spending to unanticipated government spending shocks at horizon zero with the IRF of investment to the same shock and horizon is \(-0.27\). Additionally, the penalty function approach is over-estimating the response of GDP and consumption to an unanticipated government spending shock because the correlations of the IRFs of GDP and consumption with the IRF of government spending are 0.49 and 0.28, respectively. By inducing these correlated responses the penalty function approach is introducing additional sign restrictions on these variables and, thus, not respecting the agnosticism of the identification scheme.

Let us now focus on the artificially narrow confidence intervals generated by the penalty function approach. We have shown that for each draw from the posterior distribution of the reduced-form parameters, there is a distribution of IRFs conditional on the sign and zero restrictions holding, and that the penalty function approach selects a single orthogonal matrix instead of drawing from the conditional uniform distribution. What are the consequences of this? Not surprisingly at this juncture, the consequence is artificially narrow confidence intervals. To see this, we first examine the IRFs to an unanticipated government revenue shock evaluated at the reduced-form parameters described in Figure 11. To make our point transparent, we use two separate panels to illustrate that given a value of reduced-form parameters, no uncertainty is considered when the penalty function approach is used.\(^{18}\)

\(^{18}\)In the case of optimism shocks we show in the same graph the median IRFs from the penalty function approach, and the median and 68 percent confidence intervals from the ARRW methodology to illustrate the bias present in the...
Panel (b) in Figure 11 plots the range (the interval between the maximum and the minimum IRF) of the distribution of IRFs at the OLS point estimate of the reduced-form parameters. This range characterizes the support of the distribution of IRFs consistent with the sign and zero restrictions, and it is constructed using the Cholesky decomposition of the OLS point estimate of the reduced-form parameters and several draws of the conditional uniform distribution of orthogonal matrix $Q$. The support of the IRFs reported on Panel (b) is in sharp contrast with the single IRFs reported in Panel (a) obtained using the penalty function approach evaluated at the same value of the reduced-form parameters. Figure 12 shows that the same happens in the case of the unanticipated government spending shock.

![Figure 11: IRFs to an Unanticipated Government Revenue Shock](image)

We can summarize our findings in Figure 13. The first column compares the posterior distributions of IRFs at horizon zero and the median IRFs for total government revenue, GDP, consumption, and investment to an unanticipated government revenue shock using both the penalty function approach and the ARRW methodology. In this section we abstract from replicating these graphs to highlight the fact that the penalty function approach neglects the uncertainty associated with each draw of the reduced-form parameters.
The Penalty Function Approach and the ARRW methodology. The posterior distributions are approximated using a kernel smoothing function. Comparing the penalty function approach and the ARRW methodology, we reach the following conclusions. First, the posterior distribution of IRFs of total government revenue obtained using the penalty function approach is centered around the right-hand tail of the distribution obtained using the ARRW methodology. The bias is even more clear looking at the median IRFs. Second, the penalty function approach dramatically underestimates the variance of the posterior distribution of IRFs of total government revenue. These two results were expected since the penalty function approach is maximizing the response of total government revenue in order to minimize the loss function. We also observe very narrow and downwardly biased posterior distributions of the IRFs for GDP, consumption, and investment. Column 2 does the same for the IRFs of total government spending, GDP, consumption, and investment to a unanticipated government spending shock. Again, the posterior distribution of IRFs of total government spending obtained using the penalty function approach is

\[^{19}\text{We report average median IRFs computed using horizons 0 to 3. The bias is larger using four periods and the results are emphasized. In any case, the bias basically persists using horizon 0 only.}\]
centered around the right-hand tail of the distribution obtained using the ARRW methodology. We also observe very narrow and downwardly biased posterior distributions of the IRFs for investment, and there is an upward bias for the response of GDP.

![Graphs showing IRFs for different variables](image)

Figure 13: Density of IRFs at Horizon Zero and Median of IRFs from Horizons Zero to Three

We now present additional evidence that the penalty function approach is not truly agnostic because it introduces additional sign restrictions on variables that are seemingly unrestricted. Table 8 compares the posterior probabilities that the IRFs for GDP, consumption, and investment to an unanticipated government revenue shock are negative in at least one of the first four horizons. The IRFs of the three variables are almost always negative when using the Mountford and Uhlig (2009) methodology, while it is negative only about 25 percent of the time when using the ARRW methodology. Hence,
the penalty function approach imposes additional sign restrictions on these three variables. Table 9 shows that the penalty function approach also distorts the posterior probabilities of negative IRFs for GDP and investment in response to an unanticipated government expenditure shock.

<table>
<thead>
<tr>
<th></th>
<th>The Penalty Function Approach</th>
<th>The ARRW Methodology</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std dev</td>
</tr>
<tr>
<td>GDP</td>
<td>-0.1367</td>
<td>0.0741</td>
</tr>
<tr>
<td>Consumption</td>
<td>-0.0960</td>
<td>0.0854</td>
</tr>
<tr>
<td>Non-res Investment</td>
<td>-0.3658</td>
<td>0.4704</td>
</tr>
</tbody>
</table>

Table 8: Posterior Probabilities of Negative IRFs in at Least One of the First Four Horizons: Unanticipated Government Revenue Shock

<table>
<thead>
<tr>
<th></th>
<th>The Penalty Function Approach</th>
<th>The ARRW Methodology</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std dev</td>
</tr>
<tr>
<td>GDP</td>
<td>0.0882</td>
<td>0.0573</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.0036</td>
<td>0.0523</td>
</tr>
<tr>
<td>Non-res Investment</td>
<td>-0.2157</td>
<td>0.3964</td>
</tr>
</tbody>
</table>

Table 9: Posterior Probabilities of Negative IRFs in at Least One of the First Four Horizons: Unanticipated Government Spending Shock

### 7.3 Three Fiscal Policy Scenarios

Equipped with the unanticipated fiscal policy shocks, we can analyze the three fiscal policy scenarios. We first study how the IRFs associated with the deficit-spending, the deficit-financed tax cuts, and the balanced-budget spending shocks change when using the ARRW methodology with respect to the results reported in Mountford and Uhlig (2009). Second, we do the same for the fiscal multipliers. As has been the case with unanticipated fiscal policy shocks, we just report results for GDP, consumption, and investment.

#### 7.3.1 IRFs

We begin with the deficit-spending shocks. Panel (a) in Figure 14 replicates the results in Figure 10 in Mountford and Uhlig (2009). Using the penalty function approach, one could conclude that deficit-spending shocks produce a drop in GDP (after a few periods of a small increase), consumption, and
investment (although the drop is only statistically significant for investment). However, once we use our methodology, these results disappear. There is a very wide range of IRFs that are consistent with these shocks, making very hard to say anything about the effects of deficit-spending shocks. In most cases the confidence intervals reported using the ARRW methodology are at least five times bigger than the confidence intervals reported using the penalty function approach. This means that, once we combine the unanticipated fiscal policy shocks, the confidence intervals get compounded and become even wider than before. Mountford and Uhlig’s (2009) conclusions are based on artificially narrow confidence intervals.

![Figure 14: IRFs to a Deficit-Spending Policy Shock](image)

Next, we study deficit-financed tax cut shocks. Panel (a) in Figure 15 replicates the results in Figure 11 in Mountford and Uhlig (2009). We can see that the median IRFs of GDP, consumption, real wages, and investment are positive and the tight 68 percent confidence intervals do not contain zero. Mountford and Uhlig (2009) use these results to claim that deficit-financed tax cut shocks work best to improve economic activity. On the contrary, the IRFs computed using the ARRW methodology do not provide evidence to support these findings. The median responses are negative for the first
few periods but, again, very wide confidence intervals make the interpretation of the median IRFs very hard. The upward bias and the artificially narrow confidence intervals obtained using the penalty function approach are behind Mountford and Uhlig’s (2009) conclusions.

Finally, Mountford and Uhlig (2009) study a balanced-budget spending scenario. Panel (a) in Figure 16 replicates the results reported in Figure 12 in Mountford and Uhlig (2009). As can be seen, the median IRFs of GDP, consumption, and investment are (almost always) negative and the narrow 68 percent confidence intervals do not contain zero. Again, once we consider our methodology, there is no evidence to support these results. The median responses are positive for the first few periods, but the confidence intervals are so wide that it is hard to conclude anything at all. Downward bias and artificially narrow confidence intervals are behind any conclusions implied by Mountford and Uhlig (2009).

Our methodology paints a completely different picture than the one reported in Mountford and Uhlig (2009). The biases that we find are very hard to interpret because these shocks are linear combinations of shocks that are already biased. The lack of agnosticism of the penalty function approach makes the interpretation of the median IRFs very difficult.
The Penalty Function Approach

The ARRW Methodology

Figure 16: IRFs to a Balanced-Budget Shock

The comparison between scenarios becomes even harder once we consider the cumulative discounted IRFs to either deficit-spending or deficit-financed tax cut shocks. The cumulative discounted IRFs at horizon $\tau$ of variable $y$ to the combined shock $s$ is $\sum_{t=0}^{\tau}(1 + i)^{-t}e_y L_t \left( A_0, A_+ \right) q_s$, where $e_y$ is a unit vector that selects the IRF of the variable under analysis, $q_s$ defines either the deficit-spending or deficit-financed tax cut shock depending on the value of $s \in \{DS,DTC\}$, and $i$ denotes the average real interest rate over the sample. The real interest rate is computed as the difference between the federal funds rate and the inflation rate implied by the GDP deflator and, in our sample, equals 2.51 percent — annualized.

Panel (a) in Figure 17 replicates the results reported in Figure 13 in Mountford and Uhlig (2009). The panel shows that the median cumulative discounted IRF of GDP to a deficit-spending shock becomes negative after a few periods and in the case of a deficit-financed tax cut shock is always positive. Moreover, the 68 percent confidence intervals associated with the shocks are narrow and in the case of deficit-financed tax cut shocks do not contain zero. Based on this evidence, Mountford and Uhlig
(2009) conclude that a deficit-financed tax cut scenario works best to improve GDP. Unfortunately, once we use the truly agnostic ARRW methodology, this result also disappears. The median cumulative discounted IRF of GDP to a deficit-spending shock is positive during 25 periods and it is negative for 10 periods for the case of deficit-financed tax cut shocks. As before, these biases are very hard to interpret because the deficit-spending and the deficit-financed tax cuts shocks are linear combinations of shocks that are already biased. In any case, the correctly computed 68 percent confidence intervals contain zero for both IRFs and are at least five times larger than the ones reported using the penalty function approach. Again, the lack of agnosticism of the penalty function approach is mostly reflected in extremely narrow confidence intervals.

7.3.2 Fiscal Multipliers

In addition to the IRF analysis, Mountford and Uhlig (2009) compute fiscal multipliers to compare the effects of deficit-spending shocks and deficit-financed tax cut shocks. Specifically, they compute the present value multipliers at horizon $\tau$ of the combined shock $s$ on variable $y$

$$
\frac{\sum_{t=0}^{\tau}(1+i)^{-t}e_{t}^{s}L_{t}(A_{0}, A_{+})q_{s}}{\sum_{t=0}^{\tau}(1+i)^{-t}e_{t}^{s}v_{s,y}L_{t}(A_{0}, A_{+})q_{s}} \frac{1}{(f/GDP)}
$$

and the impact multipliers at horizon $\tau$ of the combined shock $s$ on variable $y$
\[
\frac{e'y_t(A_0, A_+)(q_s - 1)}{e'f_s y_0(A_0, A_+)q_s (f/GDP)}
\]

where \(e_y\) is a unit vector that selects the IRF of the variable under analysis, \(e_f\) is a unit vector that selects the IRF of the fiscal variable (total government revenue or total government spending), \((f/GDP)\) denotes the average share of the selected fiscal variable in GDP over the sample, and \(v_{s,y}\) is equal to \(-1\) if \(s = DTC\) and \(y \neq GDP\) and equals \(1\) otherwise. The indicator variable \(v_{s,y}\) is a normalization so that the multiplier of the deficit-financed tax cut shock can be interpreted as the increase in GDP in response to a decrease in total government revenue.

In the case of present value multipliers, \(y\) is GDP and \(f\) is total government spending (revenue) when \(s = DS\) (DTC). In the case of impact multipliers, \(y\) can be GDP, total government spending, or total government revenue and \(f\) is total government spending (revenue) when \(s = DS\) (DTC).

Table 10 reports the median multipliers. We also report the 68 percent confidence intervals. Also, quarter \(t\) in the table corresponds to horizon \(t - 1\) in the above formulas.

Panel (a) shows the present value multipliers associated with deficit-spending and deficit-financed tax cut shocks. The upper block of this panel replicates the results reported in Table 2 in Mountford and Uhlig (2009). The bottom block of this panel reports the results obtained using our methodology. Using the penalty function approach, the median multipliers associated with deficit-financed tax cut shocks are positive for at least 12 quarters. In contrast, these median multipliers are negative at all horizons using the ARRW methodology. When we consider deficit-spending shocks, we find that while the median multipliers are negative after 12 quarters using the penalty function approach, they are positive during 20 quarters using the ARRW methodology. Also, the median multipliers associated with the deficit-spending shocks are larger. At their maximum value, the multipliers of the deficit-spending shocks are five times larger than the ones reported using the penalty function approach. As already mentioned, these biases are hard to interpret because the multipliers being analyzed correspond to shocks that are linear combinations of shocks that are already biased. Most important, the ARRW methodology reports confidence intervals that are huge relative to the ones obtained using the penalty function approach.

Panel (b) presents the impact multipliers associated with deficit-financed tax cut shocks. The upper block of this panel replicates the results reported in Table 3 in Mountford and Uhlig (2009). The bottom block of the panel reports the results obtained using the ARRW methodology. While Mountford and Uhlig (2009) find positive GDP median multipliers for at least 20 quarters, we find
<table>
<thead>
<tr>
<th></th>
<th>1 qrt</th>
<th>4 qrtS</th>
<th>8 qrtS</th>
<th>12 qrtS</th>
<th>20 qrtS</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(a) Present Value Multipliers</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>The Penalty Function Approach</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DTC</td>
<td>0.30</td>
<td>0.53</td>
<td>1.62</td>
<td>4.79</td>
<td>-4.71</td>
<td>4.79</td>
</tr>
<tr>
<td></td>
<td>[0.13 , 0.50]</td>
<td>[0.44 , 0.63]</td>
<td>[1.15 , 2.27]</td>
<td>[0.80 , 13.93]</td>
<td>[-11.34 , 2.93]</td>
<td>(qrt 12)</td>
</tr>
<tr>
<td>DS</td>
<td>0.62</td>
<td>0.45</td>
<td>0.05</td>
<td>-0.30</td>
<td>-1.45</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>[0.26 , 0.99]</td>
<td>[0.33 , 0.57]</td>
<td>[-0.30 , 0.43]</td>
<td>[-1.05 , 0.41]</td>
<td>[-4.87 , 0.34]</td>
<td>(qrt 1)</td>
</tr>
<tr>
<td><strong>The ARRW Methodology</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DTC</td>
<td>-0.75</td>
<td>-0.73</td>
<td>-1.04</td>
<td>-1.56</td>
<td>-1.83</td>
<td>-0.73</td>
</tr>
<tr>
<td></td>
<td>[-2.88 , 1.19]</td>
<td>[-2.80 , 3.00]</td>
<td>[-3.98 , 2.19]</td>
<td>[-4.86 , 1.56]</td>
<td>[-3.54 , 0.03]</td>
<td>(qrt 4)</td>
</tr>
<tr>
<td>DS</td>
<td>1.12</td>
<td>1.03</td>
<td>1.11</td>
<td>1.58</td>
<td>2.99</td>
<td>3.79</td>
</tr>
<tr>
<td><strong>(b) Impact Multipliers: DTC</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>The Penalty Function Approach</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP</td>
<td>0.30</td>
<td>0.95</td>
<td>2.05</td>
<td>3.23</td>
<td>2.33</td>
<td>3.38</td>
</tr>
<tr>
<td></td>
<td>[0.13 , 0.50]</td>
<td>[0.77 , 1.13]</td>
<td>[1.31 , 2.85]</td>
<td>[2.09 , 4.59]</td>
<td>[0.58 , 4.32]</td>
<td>(qrt 14)</td>
</tr>
<tr>
<td>Total Gov Revenue</td>
<td>-1.00</td>
<td>-1.00</td>
<td>0.02</td>
<td>0.91</td>
<td>0.89</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-1.00 , -1.00]</td>
<td>[-1.00 , -1.00]</td>
<td>[-0.33 , 0.38]</td>
<td>[0.36 , 1.61]</td>
<td>[-0.05 , 1.93]</td>
<td></td>
</tr>
<tr>
<td>Total Gov Spending</td>
<td>0.00</td>
<td>0.00</td>
<td>0.28</td>
<td>0.44</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-0.00 , 0.00]</td>
<td>[-0.00 , 0.00]</td>
<td>[0.18 , 0.38]</td>
<td>[0.26 , 0.61]</td>
<td>[0.13 , 0.74]</td>
<td></td>
</tr>
<tr>
<td><strong>The ARRW Methodology</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP</td>
<td>-0.75</td>
<td>-0.42</td>
<td>1.08</td>
<td>1.10</td>
<td>0.19</td>
<td>1.18</td>
</tr>
<tr>
<td>Total Gov Revenue</td>
<td>-1.00</td>
<td>-1.00</td>
<td>0.07</td>
<td>0.20</td>
<td>0.09</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-1.00 , -1.00]</td>
<td>[-1.00 , -1.00]</td>
<td>[-1.74 , 6.75]</td>
<td>[-4.75 , 5.65]</td>
<td>[-6.04 , 7.85]</td>
<td></td>
</tr>
<tr>
<td>Total Gov Spending</td>
<td>0.00</td>
<td>0.00</td>
<td>0.13</td>
<td>0.12</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-0.00 , 0.00]</td>
<td>[-0.00 , 0.00]</td>
<td>[-0.51 , 1.39]</td>
<td>[-0.68 , 3.14]</td>
<td>[-1.26 , 2.70]</td>
<td></td>
</tr>
<tr>
<td><strong>(c) Impact Multipliers: DS</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>The Penalty Function Approach</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP</td>
<td>0.62</td>
<td>0.28</td>
<td>-0.73</td>
<td>-0.98</td>
<td>-1.70</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>[0.26 , 0.99]</td>
<td>[-0.12 , 0.70]</td>
<td>[-1.70 , 0.29]</td>
<td>[-2.45 , 0.30]</td>
<td>[-3.97 , 0.18]</td>
<td>(qrt 1)</td>
</tr>
<tr>
<td>Total Gov Spending</td>
<td>1.00</td>
<td>1.00</td>
<td>0.35</td>
<td>-0.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[1.00 , 1.00]</td>
<td>[1.00 , 1.00]</td>
<td>[0.63 , 0.97]</td>
<td>[0.10 , 0.62]</td>
<td>[-0.71 , 0.19]</td>
<td></td>
</tr>
<tr>
<td>Total Gov Revenue</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.38</td>
<td>-0.84</td>
<td>-1.77</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-0.00 , 0.00]</td>
<td>[-0.00 , 0.00]</td>
<td>[-0.93 , 0.22]</td>
<td>[-1.65 , 0.03]</td>
<td>[-3.24 , -0.42]</td>
<td></td>
</tr>
<tr>
<td><strong>The ARRW Methodology</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP</td>
<td>1.12</td>
<td>0.46</td>
<td>-0.01</td>
<td>0.02</td>
<td>-0.50</td>
<td>1.46</td>
</tr>
<tr>
<td>Total Gov Spending</td>
<td>1.00</td>
<td>1.00</td>
<td>0.72</td>
<td>0.28</td>
<td>-0.14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[1.00 , 1.00]</td>
<td>[1.00 , 1.00]</td>
<td>[-4.54 , 2.20]</td>
<td>[-5.98 , 2.85]</td>
<td>[-6.09 , 4.53]</td>
<td></td>
</tr>
<tr>
<td>Total Gov Revenue</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>-0.46</td>
<td>-1.16</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-0.00 , 0.00]</td>
<td>[-0.00 , 0.00]</td>
<td>[-8.82 , 10.13]</td>
<td>[-14.48 , 13.97]</td>
<td>[-17.06 , 16.06]</td>
<td></td>
</tr>
</tbody>
</table>

Table 10: Fiscal Multipliers

negative ones during the four quarters following the initial shock. After four quarters the median multipliers associated with the ARRW methodology also become positive. In addition, even when they share a sign, the median multipliers associated with our methodology are much smaller than the median multipliers implied by the penalty function approach. In any case, as before, the confidence intervals computed using the ARRW methodology are so large that it very hard to say anything concrete about the sign and size of the multipliers.

Panel (c) presents the impact multipliers associated with deficit-spending shocks. The upper block of this panel replicates the results reported in Table 4 in Mountford and Uhlig (2009), and the bottom block of the panel reports the results obtained using the ARRW methodology. In this policy scenario,
both methodologies find the same sign (except for the 12 quarters) for the median multiplier. However, the magnitudes are different. The absolute value of the GDP median multipliers resulting from the ARRW methodology is approximately twice as large as the one resulting from the penalty function approach. But it is also the case that the confidence intervals computed using the ARRW methodology are so wide that it is very hard to reach any conclusion.

Summarizing, Mountford and Uhlig (2009) use their results regarding the fiscal multipliers to emphasize that deficit-financed tax cut shocks work best to increase economic activity. But as mentioned before the penalty function approach is not agnostic. Once we use our truly agnostic methodology, it is very hard to support Mountford and Uhlig’s (2009) claims. Some median multipliers change sign; nevertheless, the correct confidence intervals are so wide that it is very hard to reach any conclusion from a statistical point of view.

### 7.4 Computational Time

Our methodology is also faster than the penalty function approach in the case of unanticipated fiscal policy shocks. In this case, the penalty function approach is approximately three times slower than our methodology. The results are closer, but the ARRW methodology is still faster. In order to avoid local minima when using the penalty function approach, we start the non-linear optimization from eight random starting points and then we pick the best one. Mountford and Uhlig (2009) follow a similar strategy.

<table>
<thead>
<tr>
<th></th>
<th>The Penalty Function Approach</th>
<th>The ARRW Methodology</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiscal Policy Shocks</td>
<td>12498.06</td>
<td>3929.3</td>
<td>3.18</td>
</tr>
</tbody>
</table>

Table 11: Computational Time in Seconds

### 8 Conclusion

We have presented an efficient algorithm for inference in SVARs identified with sign and zero restrictions that properly draws from the posterior distribution of structural parameters. The algorithm extends the sign restrictions methodology developed by Rubio-Ramírez et al. (2010) to allow for zero restrictions. Our key theoretical contribution shows how to efficiently draw from the uniform distribution with respect to the Haar measure on the set of orthogonal matrices conditional on some linear
restrictions on their coefficients holding. This is the crucial step that allows us to draw from the posterior distribution of structural parameters conditional on the sign and zero restrictions. We have used this algorithm to answer the following questions. Are optimism shocks an important source of business cycle fluctuations? Are deficit-financed tax cuts better than deficit spending to increase output? These questions have been previously studied by Beaudry et al. (2011) and Mountford and Uhlig (2009), respectively, using the penalty function approach. These authors have provided very definitive answers. Unfortunately, we have shown that these sharp conclusions are due to shortcomings in the penalty function approach. In particular, we have shown that the penalty function approach (1) imposes additional sign restrictions on variables that are seemingly unrestricted that bias the results, and (2) it chooses a single value of structural parameters, instead of drawing from its posterior, creating artificially narrow confidence intervals that also affect inference and the economic interpretation of the results. These shortcomings appear because the penalty function approach does not correctly draw from the posterior distribution of structural parameters conditional on the sign and zero restrictions. This problem is common to all of the existing methods. Our algorithm is also faster than the current methods.
9 Appendices

9.1 Appendix A. Estimation and Inference: Optimism Shocks

Following Beaudry et al. (2011) we estimate equation (3) with four lags using Bayesian methods with a Normal-Wishart prior as in Uhlig (2005). Specifically, we take 1,000 parameter draws from the Normal-Wishart posterior of the reduced-form parameters \((\mathbf{B}, \Sigma)\) and from the conditional uniform distribution of \(Q\). We use the data set created by Beaudry et al. (2011). This data set contains quarterly U.S. data for the sample period 1955Q1-2010Q4 and includes the following variables: TFP, stock price, consumption, real federal funds rate, hours worked, investment, and output. TFP is the factor-utilization-adjusted TFP series from John Fernald’s website. Stock price is the Standard and Poor’s 500 composite index divided by the CPI of all items from the Bureau of Labor Statistics (BLS). Consumption is real consumption spending on non-durable goods and services from the Bureau of Economic Analysis (BEA). The real federal funds rate corresponds to the effective federal funds rate minus the inflation rate as measured by the growth rate of the CPI all items from the BLS. Hours worked is the hours of all persons in the non-farm business sector from the BLS. Investment is real gross private domestic investment from the BEA. Output is real output in the non-farm business sector from the BLS. The series corresponding to stock price, consumption, hours worked, investment, and output are normalized by the civilian non-institutional population of 16 years and over from the BLS. All variables are logarithmic levels except for the real interest rate that is in levels but not logged.

9.2 Appendix B. Estimation and Inference: Fiscal Policy Shocks

Following Mountford and Uhlig (2009) we estimate equation (3) with six lags using Bayesian methods with a Normal-Wishart prior as specified in Uhlig (2005). We take 1,000 parameter draws from the Normal-Wishart posterior \((\mathbf{B}, \Sigma)\) and from the conditional uniform distribution of \(Q\). We use the same data set as Mountford and Uhlig (2009). This data set contains quarterly U.S. data for the sample period 1955Q1-2010Q4 and includes the following variables: GDP, private consumption, total government spending, total government revenue, real wages, private non-residential investment, interest rate, adjusted reserves, producer price index of raw materials, and GDP deflator. All variables are logarithmic levels except for the interest rate that is expressed in levels but it is not logged.

9.3 Appendix C. Tables and Figures
The ARRW Methodology

The Penalty Function Approach

Table 12: Share of FEV Attributable to Optimism Shocks. Five Variable SVAR
The ARRW Methodology

<table>
<thead>
<tr>
<th>Identification 1</th>
<th>Identification 2</th>
<th>Identification 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adjusted TFP</td>
<td>0.08</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>[0.03, 0.21]</td>
<td>[0.03, 0.23]</td>
</tr>
<tr>
<td>Stock Price</td>
<td>0.12</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>[0.03, 0.34]</td>
<td>[0.04, 0.39]</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.11</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>[0.02, 0.35]</td>
<td>[0.03, 0.40]</td>
</tr>
<tr>
<td>Real Interest Rate</td>
<td>0.13</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>[0.06, 0.27]</td>
<td>[0.05, 0.27]</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>0.12</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>[0.03, 0.33]</td>
<td>[0.04, 0.34]</td>
</tr>
<tr>
<td>Investment</td>
<td>0.13</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>[0.05, 0.30]</td>
<td>[0.05, 0.35]</td>
</tr>
<tr>
<td>Output</td>
<td>0.12</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>[0.03, 0.33]</td>
<td>[0.05, 0.39]</td>
</tr>
<tr>
<td>Labor Productivity</td>
<td>0.10</td>
<td>0.12</td>
</tr>
<tr>
<td></td>
<td>[0.02, 0.29]</td>
<td>[0.03, 0.34]</td>
</tr>
</tbody>
</table>

The Penalty Function Approach

<table>
<thead>
<tr>
<th>Identification 1</th>
<th>Identification 2</th>
<th>Identification 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adjusted TFP</td>
<td>0.17</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>[0.08, 0.30]</td>
<td>[0.08, 0.38]</td>
</tr>
<tr>
<td>Stock Price</td>
<td>0.52</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>[0.34, 0.70]</td>
<td>[0.48, 0.73]</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.13</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td>[0.06, 0.28]</td>
<td>[0.39, 0.69]</td>
</tr>
<tr>
<td>Real Interest Rate</td>
<td>0.10</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>[0.05, 0.17]</td>
<td>[0.03, 0.13]</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>0.16</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>[0.09, 0.26]</td>
<td>[0.25, 0.53]</td>
</tr>
<tr>
<td>Investment</td>
<td>0.25</td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td>[0.16, 0.37]</td>
<td>[0.34, 0.57]</td>
</tr>
<tr>
<td>Output</td>
<td>0.23</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td>[0.13, 0.39]</td>
<td>[0.46, 0.72]</td>
</tr>
<tr>
<td>Labor Productivity</td>
<td>0.24</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>[0.12, 0.39]</td>
<td>[0.17, 0.55]</td>
</tr>
</tbody>
</table>

Table 13: Share of FEV Attributable to Optimism Shocks at Horizon 40. Seven Variable SVAR.
Table 14: Share of FEV Attributable to Optimism Shocks: Seven-Variable SVAR
Figure 18: IRFs to an Optimism Shock Using the Penalty Function Approach: Seven-Variable SVAR

(a) Identification 1  
(b) Identification 2  
(c) Identification 3
Figure 19: IRFs to an Optimism Shock Using the ARRW Methodology: Seven-Variable SVAR
Figure 20: Comparison of IRFs to an Optimism Shock: Seven-Variable SVAR

Note: Median PFA refers to the median IRF obtained using the penalty function approach.
Figure 21: Distribution of IRFs with the ARRW methodology vs Single IRFs with the Penalty Function Approach: Seven-Variable SVAR

Note: OLS PFA refers to the IRF obtained using the penalty function approach and the OLS reduced-form estimates.
Figure 22: Replicating the Penalty Function Approach using the ARRW Methodology: Five-Variable SVAR
References


2013-21: "Do Spanish informal caregivers come to the rescue of dependent people with formal care unmet needs?", Sergi Jiménez-Martín y Cristina Vilaplana Prieto.
2013-17: "Rainfall Risk and Religious Membership in the Late Nineteenth-Century US", Philipp Ager y Antonio Ciccone.