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Abstract

We investigate the trade-off, arising in financial networks, between higher risk-sharing and greater exposure to contagion when the connectivity increases. We find that with shock distributions displaying "fat" tails, extreme segmentation into small components is optimal, while minimal segmentation and high density of connections are optimal with distributions exhibiting "thin" tails. For less regular distributions, intermediate degrees of segmentation and sparser connections are optimal. If firms are heterogeneous, optimality requires perfect assortativity in their linkages. In general, however, a conflict arises between optimality and individual incentives to establish linkages, due to a "size externality" not internalized by firms.

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Risk-sharing and contagion in networks

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En este artículo analizamos un modelo en el que las empresas hacen frente a un choque aleatorio que hace que pueda ser incapaz de pagar las obligaciones derivadas de los pasivos y debe suspender pagos, lo que supone una pérdida de los retornos futuros que esta inversión podría generar. En este contexto las empresas se pueden beneficiar de entrar en acuerdos de intercambio de activos con otras empresas para diversificar riesgos. Este intercambio hace que el choque recibido por una empresa sea soportado por ésta solamente de manera parcial. A su vez, esto genera la exposición de una empresa a los choques recibidos por otras.

El artículo estudia primero cómo depende el bienestar social de la estructura de la red, con particular atención a dos dimensiones: el tamaño de los grupos de empresas, y la densidad relativa de conexiones en esos grupos. A grandes rasgos, cuando la distribución de probabilidad coloca una masa de probabilidad lo suficientemente alta en valores pequeños de los choques ("colas delgadas"), la mejor configuración es que todas las empresas pertenezcan a un único grupo bien conectado. El objetivo principal en este caso es conseguir el mayor nivel de riesgo compartido. En el caso opuesto, en el que la distribución de probabilidad de los choques exhibe "colas gruesas" (es decir, hay una gran masa de probabilidad concentrada en choques grandes), la configuración óptima implica un grado «máximo de segmentación. Esto refleja una situación en la que la prioridad es reducir al mínimo el contagio.

Estos dos casos extremos, sin embargo, no agotan todas las posibilidades. Para especificaciones más complejas de la estructura de choque (por ejemplo, mezclas de colas delgadas y gruesas) son óptimas configuraciones intermedias. En cuanto a la heterogeneidad, la principal conclusión es que ésta tiende a favorecer que las empresas que se enfrentan las distribuciones de choques similares deben agruparse. Eso significa, en la práctica, que algunas actividades deben ser aisladas de otras; por ejemplo, separando las actividades de banca comercial de las de banca de inversión.

Es habitual que exista un conflicto entre la eficiencia y las decisiones individuales en problemas de formación de red. Y por esto el artículo estudia también si los requisitos para la optimalidad de la estructura de red del sistema son compatibles con los incentivos de las empresas individuales para establecer enlaces. En este caso se analiza la cuestión mediante el examen de los equilibrios a prueba de coaliciones (CPE, por sus siglas en inglés), en la que cualquier grupo (es decir, cualquier coalición) de empresas se puede desviar de formación conjunta. En este caso no hay problemas dentro de cada grupo específico, donde los incentivos individuales se conjugan bien con los del grupo total, porque todos tienen las mismas preferencias respecto al tamaño. La dificultad en este caso viene de que una vez alcanzado el tamaño óptimo desde el punto de vista del grupo, sus miembros no tienen en cuenta que la reducción total del riesgo por invitar a un miembro más puede ser mayor que la suma de los impactos en los miembros actuales del grupo. Y, por tanto, en general los CPE's tienen empresas que se encuentran en componentes ineficientemente pequeñas.

1 Introduction

Recent economic events have made it clear that looking at financial entities in isolation, abstracting from their linkages, gives an incomplete, and possibly very misleading, impression of the potential impact of shocks to the financial system. In the words of Acharya *et al.* (2010) "current financial regulations, such as Basel I and Basel II, are designed to limit each institution's risk seen in isolation; they are not sufficiently focused on systemic risk even though systemic risk is often the rationale provided for such regulation." The aim of this paper is precisely to investigate how the capacity of the system to absorb shocks depends on the pattern of interconnections established among firms, say banks.

More specifically, we intend to study the extent to which the risk-sharing benefits to firms of becoming more highly interconnected (i.e. to gain some insurance against relatively small shocks, absorbable within the whole system) may be offset by larger costs resulting from an increased risk exposure (which, for large shocks, could entail a large wave of induced bankruptcies). That is, we want to analyze the trade-off between risk-sharing and contagion. Clearly, this trade-off must be at the center of any regulatory efforts of the financial world that takes a truly systemic view of the problem. This paper highlights some of the considerations that should play a key role in this endeavor. In particular, by formulating the problem in a stylized and analytically tractable framework, it examines how the segmentation of the system into separate components, the density of the connections within each component, as well as asymmetries in the pattern of connections, should be tailored to the underlying shock structure. It also sheds light on the key issue of whether the normative prescriptions on the optimal pattern of linkages are consistent with the individual incentives to form or remove links.

We analyze an environment with N financial firms and a continuum of small investors. For simplicity, in most of the paper we shall consider the case where all firms are *ex ante* identical, with the same level of assets and liabilities, and endowed with a risky project displaying the same probabilistic pattern of returns. But *ex post* they will be different since we assume that, with some probability, a shock hits a randomly selected firm, decreasing the income generated by the firm's project, thus possibly leading to the default of that firm if the resulting income from the firm's assets falls short of its liabilities. The precise effects of the shock depend on its pattern of financial linkages among the firms. The more the firm is connected to other firms the smaller will be the effect on the firm's own assets, but then these other firms will also be affected by the shock. By way of illustration, we can interpret the linkages among firms as originating from an exchange of assets among them. For example, think of the securitization of mortgage loans, which are then sold to investors different from the originator. Those investors can, in turn, sell claims to their own portfolios and thus generate further diversification for themselves, but this also makes the buyers of such claims exposed to the returns of the original mortgage loans. In the end, the overall network of connections generates patterns of mutual exposure between any pair of connected firms. Consequently, when a shock hits a firm, any other firm connected with it becomes affected in proportion to its exposure to that firm and has to default when its share of the shock exceeds the value of the firm's assets, net of its liabilities.

In this way, the exposure to losses in commonly held assets induces a sort of contagion, the process of asset exchange in our model being analogous to the transmission of pathogens in disease contagion. An asset that turns "bad" (for example a loan to a developer that is unpaid) exposes all common owners of that asset to a shock. So when the shock hits, it is transmitted from the originator bank to others who purchased its mortgage-backed securities, and further down the line to the companies which purchased the securities issued by those latter firms. To render the contagion analogy precise, we must conceive the "disease" in our case as one that can be transmitted well before symptoms are present (AIDS is a good example of such a disease). It must also have the feature that not all infected firms must die, for in our context what causes the default of one firm need not lead to the default of another. In the end, it is the *overall* 'network' structure of the financial system that determines how any given shock affects different firms and what is its overall aggregate impact on the whole system.

Clearly, the maximum extent of risk sharing obtains when all firms belong to a single and fully connected network. This configuration, however, exhibits the widest exposure of firms in the system to shocks and any shock must affect all firms in the system and, if large, could lead to extensive default. There are two alternative (and in some cases complementary) ways of reducing such exposure. One is by 'segmentation, which isolates the firms in each component from the shocks that hit any other component.[1] The second route to lowering exposure is by making firms *directly* connected to only a subset of the other firms in their component, while having exposure decline in the network distance between firms within the component. In fact, to obtain a sharp understanding of the effect of network distance on contagion, our analysis will contrast two cases: (i) completely connected components, where there is a direct link between any pair of firms in each component, and therefore the mutual exposure between any pair of firms is exactly the same; (ii) ring components, where the reciprocal exposure between firms is heterogeneous, falling with network distance.

The key objective of the paper is to identify the architecture of the system that best tackles the trade-off between risk sharing and contagion, hence minimizing the expected number of defaults in the system. Our model captures the essence of the problem and allows the study of such a trade-off under a fairly general structure of the random shocks. In particular, as this shock structure changes, we obtain a precise determination of how the *optimal degree of segmentation* as well as *optimal link density* correspondingly adapt.

A first set of our results can be summarized as follows. We find that when the probability distribution of the shocks exhibits "fat tails" (i.e. attributes a high mass to large shocks), the optimal configuration involves a maximum degree of segmentation – that is, components should be of the minimum possible size. This reflects a situation where the priority is to minimize contagion. Instead, in the opposite case where the probability distribution places high enough mass on relatively small shocks ("thin tails), the best configuration has all firms arranged in a single component. The main aim in this latter case is to achieve the highest level of risk sharing. These two polar cases, however, do not exhaust all possibilities. For we also find that for other, more complex specifications of the shock structure (e.g. mixtures of fat and thin tails) intermediate arrangements are optimal, i.e. the optimal degree of segmentation may involve medium-sized components.

It is interesting to note that all of the previous conclusions apply irrespectively of whether one considers structures involving completely connected components or, alternatively, minimally connected ones. As explained, the potential advantage of minimally connected struc-

[1] A component of the network is defined as a maximal set of firms that are directly or indirectly connected.

tures is that the exposure between firms in a component is not uniform but decays with network distance, which limits the number of defaults induced when the shocks are large. A natural implication of this feature is that, among minimally connected structures, the optimal degree of segmentation is lower (or, equivalently, the component size is larger) than for completely connected ones. This conclusion notwithstanding, we find that when segmentation in complete components is implemented optimally, segmentation in minimally connected structures always dominates if the shock distributions are as described in the previous paragraph. It is however important to emphasize that for other, less regular shock distributions a minimally connected structure is optimal (i.e. better than any segmentation in complete components). This happens, for instance, when the shock distribution assigns a high probability mass on a small range of large shocks as well as on a wide range of small and medium size shocks. In those cases, we find that low density is preferred to more segmentation as the mechanism for limiting contagion.

The paper also addresses the issue of whether the requirements for optimality are compatible with the incentives of individual firms to establish links. Formally, we analyze this issue by examining the *Coalition-Proof Equilibria* (CPE) of a network formation game, where any group (i.e. coalition) of firms can jointly deviate[2]. Our main conclusion is that there is typically a conflict between efficiency and individual incentives. This conflict derives from the fact that CPE typically exhibit heterogeneities in component size and these are inconsistent with efficiency. There are, therefore, positive externalities associated to displaying a uniform size for all components that are not internalized at CPE – in general, the equilibria have some firms that lie in a component that is inefficiently small.

The results summarized so far refer to environments where all firms are *ex ante* identical and thus operate at the same scale and face the same shock distribution. The paper, however, also studies the case where firms may be different, either with regard to their size or to their shock distribution. Under these circumstances, our main conclusion is particularly sharp: if we focus on completely connected components, optimality requires perfect assortativity. That is, any optimal configuration must have all firms arranged in

[2] This is used to avoid the multiplicity issues associated to the coordination problems present in this kind of games under more standard equilibrium notions.

homogeneous components (in each of them, all firms are identical both in size and in shock distributions).

Finally, we also consider the implications of allowing for asymmetric network structures and find that the network positioning of firms within a component matters. In particular, we contrast two situations: the symmetric configuration where components are completely connected (and hence all firms are in a symmetric position), and the case where firms are arranged in star networks (thus there is a large central firm connected to other smaller peripheral ones). We find that the symmetric structure is optimal when the shocks are not too large (because this maximizes risk-sharing possibilities) while the star structure is optimal for bigger shocks. The latter conclusion reflects the fact that star networks limit contagion when shocks are large, by restricting overall connectivity and having the central larger nodes act as buffers.

Our model, although stylized, allows for the guidance of concrete policy advice, particularly in light of the robustness of many of our implications found by Loepfe et al. (2013).[3] As mentioned, we find that one cannot generally expect that social and individual incentives be aligned. But there is more than that, in particular concerning how actual policy should deal with the issue of systemic risk. In view of the increasing trend of financial markets integration that has substantially increased firms' interconnections (see Diebold and Yilmaz 2009), it is now widely judged to be insufficient to rely primarily on the usual approach to the problem, i.e. the regulation of capital requirements.[4] But even when network characteristics have entered the policy discussion, they have done so by focusing attention on institutions that are "too big to fail"– or in the more enlightened versions, "too central to fail," as in Battiston et al. (2012). Our paper, however, makes it clear that the derivation of an optimal structure goes far beyond understanding which are the most systemic institutions, because even in a model with symmetric firms, the structure of the network plays a very important role for the stability of the system. In particular, the optimal structure crucially depends on the nature of the shocks affecting the firms in the system. A clear

[3] We postpone a summary of the robustness exercise conducted in Lopefe *et al.* (2013) to the concluding Section 6, once the analysis of the present paper is complete.

[4] The Basel III agreements require higher overall equity requirements than previous rounds plus and additional capital conservation buffer of 2.5% during expansions to combat the pro-cyclicality of credit.

recommendation that emerges from our paper is that as the distribution puts more weight on larger shocks, the degree of interconnection should become smaller.

Another important aspect we have highlighted is that when different institutions are hit by shocks coming from different distributions, they should belong to separate components. This result could possibly add a novel justification for the separation of investment and commercial banking which was enforced by the Glass-Steagall act from 1933 to 1999. The traditional view was that the separation is necessary to protect consumers from conflict of interest between these institutions, but Kroszner and Rajan (1994), for example, find little evidence for this argument. In our model, the separation may prove beneficial as a way to cope with different types of shocks hitting alternative types of institutions. It is conceivable that, for this reason, commercial banks should have a much larger component size than investment banks.

We must nevertheless be cautious in interpreting our policy implications. The financial network structures we observe in reality are rather different from the ones we identify as optimal in this paper. They are often polarized, with some firms being far more central and well-connected than others. It would be premature to argue that this is not optimal, since we ignore forces – intermediation, for instance – that, beyond risk-sharing and limiting contagion, also contribute to determining the form of these networks . It may be argued, however, that the elements highlighted in this paper are quite relevant and important, and should be taken into account by policy makers in shaping their decisions.

1.1 Related literature

We end this introduction with a brief review of the related literature[5]. The research on financial contagion and systemic risk is quite diverse and also fast-growing. Hence we shall provide here only a brief summary of some of the more closely related papers.[6]

Allen and Gale (2000) pioneered the study of the stability of interconnected financial

[5]The reader is referred to Allen and Babus (2009) and to Cabrales et al. (2014) for recent surveys of the literature on financial contagion in networks.

[6]There is also a large body of literature that studies the general problem of risk sharing in non financial contexts, largely motivated by its application to consumption sharing in poor economies that lack formal insurance mechanisms. Paradigmatic examples are the papers by Bramoullé and Kranton (2007), Bloch *et al.* (2008), and Ambrus *et al.* (2011).

systems. They analyze a model in the Diamond and Dybvig (1983) tradition, where a single, completely connected component is *always* the efficient network structure, i.e. the one that minimizes the extent of default. Our model, in contrast, shows that a richer shock structure can generate a genuine trade-off between risk-sharing and contagion and that in some circumstances a certain degree of segmentation and/or low density may turn out to be efficient.

Freixas *et al.* (2000) consider an environment where a *lower density* of interaction, even though it limits risk sharing, has the positive consequence of reducing the incentives for deposit withdrawal. Concerning *segmentation*, on the other hand, a positive role for it is obtained by Leitner (2005) via numerical analysis in a model where, unlike ours, no role is played by the pattern of linkages among firms in a component, since risk is always shared completely within each component.

More in line with our approach, Allen et al. (2011) consider a six-firm environment where each firm needs funds for its investment. Since these investments are risky, firms may gain from risk diversification, which is achieved by exchanging shares with other firms. This gives rise to a financial network for which two possibilities are considered: a segmented and an unsegmented structure. The paper then analyzes the different effects in these two structures of the arrival of a signal indicating that *some* firm in the system will have to default. On a similar line, Caballero and Simsek (2013) and Alvarez and Barlevy (2014) have studied how informational contagion can amplify the contagion resulting from mutual exposure between firms.

Three more recent, related papers are Elliott et al. (2013), Acemoglu et al. (2013), and Glasserman and Young (2013). As explained more in detail in Remark 1 in the next section, the nature and form of the financial linkages among firms considered in these papers are partly different, as they may concern not only the asset side but also the liability side of the firms' balance sheet, and entail the presence of some amplification mechanism of the shocks hitting a firm. Our specification yields a more tractable model of contagion from the shocks hitting one firm. More importantly, as shown in Cabrales et al. (2014), the characterization of the pattern of contagion across different network structure for different sizes of the shocks hitting a firm is similar in the four models.

Another difference with our paper concerns the main focus of the analysis. While the aim of Elliott et al. (2013) is to characterize conditions on the structure of the network under which default cascades occur, the primary objective of ours is to characterize the optimal financial structures in diverse scenarios and their consistency with individual incentives. Acemoglu *et al* (2013), shares with the present one its concern with the optimality of financial networks, but focuses attention on shock distributions that are degenerate Dirac measures concentrated on a given shock magnitude. One of the main results of their paper is that, depending on whether the magnitude of shocks is small or large, the optimal network is either complete or has components almost isolated from the rest. In contrast we characterize the optimal network structure for a non trivial distribution function of the shocks, finding a richer pattern of optimal structures, which may exhibit intermediate degrees of segmentation or different levels of link density. Glasserman and Young (2013) shares some features with Acemoglu et al. (2013). An important difference is that the authors do not concentrate on fixed shocks to particular nodes as Acemoglu *et al.* (2013) do, but they rather specify a distribution that generates the shocks (as we also do). They then compute general bounds, for a variety of distributions, on the probability of default for any set of nodes. Their main result is a characterization of the parameter values and networks for which contagion is weak, meaning that default from a shock hitting a set of firms directly is larger than the one that arises from shocks hitting them indirectly. This contrasts with our approach, which characterizes optimal networks as a function of the distribution of shocks.

There is also a complementary line of literature that, in contrast with the papers just mentioned, studies the issue of contagion and systemic risk in the context of large networks (usually, randomly generated). In many of these papers, the approach is numerical, based on large-scale simulations (see e.g. Nier *et al.* (2007)). A notable exception is the recent paper by Blume *et al.* (2011), which integrates the mathematical theory of random networks with the strategic analysis of network formation. Its primary aim is to address the question of what is the socially optimal connectivity of the system (i.e. the average degree of the network) and whether such optimal connectivity is consistent with agents' incentives to create or destroy links. They find that social optimality is attained around the threshold where a large component emerges, but individuals will generally want to connect beyond this point. As in our paper, this induces a conflict between social and individual optimality, which in their case is due to the fact that agents do not internalize the effect on others of adding new channels (i.e. links) that facilitate the spread of the effects of the shocks.

Finally, we should refer to the large empirical and policy-oriented literature whose main objective has been to devise summary measures of the network of inter-firm (mostly banks) relationships that is able to anticipate systemic failures. For example, Battiston *et al.* (2012) propose a measure of centrality (*Debtrank*) that is inspired on the measure of *Pagerank* centrality that Google uses to rank web-pages, while Denbee *et al.* (2011) propose a measure of centrality á la Katz-Bonacich (following Ballester, Calvó-Armengol and Zenou (2006)) and apply it to English data. Of particular interest in this respect is Elsinger, Lehar and Summer (2011) who, using Austrian data, show that correlation in banks' asset portfolios is the main source of systemic risk. A related emphasis on cross-ownership and investment has been pursued as well by the literature on "balance sheet effects" to understand the Asian financial turmoil in the late 90's (Krugman 1999) as well as the current crisis (Ahrend and Goujard 2011). These latter approaches are very much in line with our model, which precisely highlights portfolio correlation as the key driver of default risk.

The rest of the paper is organized as follows. Section 2 presents the environment and the different types of financial networks that are considered. Section 3 characterizes optimal financial structures for various properties of the shock distribution. Section 4 addresses the network formation problem and studies the tension between strategic stability and optimality. Section 5 considers different extensions to environments with asymmetries, in particular concerning firm size, the distribution of shocks, or the underlying network architecture. Finally, Section 6 concludes with a summary and an outline of pending research issues. For convenience, the proofs of our results are relegated to the Appendix.

2 The Model

2.1 The Environment

As an initial benchmark scenario, we consider an environment with N ex ante identical financial firms (say, banks) and a continuum of small investors. Each firm i has an investment opportunity - a project - which requires a unit investment and yields a random gross return $\widetilde{R}_i$: with probability $1-q$ the return equals its 'normal' value R, but with probability q the firm's investment is hit by a shock and its return equals $R-\widetilde{L}_i$, where $\widetilde{L}_i$ is a positive valued random variable, with the same distribution for all i. The resources needed to undertake the project are obtained by issuing liabilities (e.g. deposits or bonds) on which a deterministic gross return M_i must be paid.

Financial Linkages We model financial linkages by assuming that starting from an original situation where every firm holds all of the claims on its own project, each of them can end up owning claims to the yield of the investment of other firms. More precisely, for any $i, j = 1, .., N$ we denote by $a_{ij} \geq 0$ the fraction held by firm i of the outstanding amount of claims to the yield of the project of firm j. Hence the pattern of asset holdings across the N firms in the economy is described by a matrix A of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \tag{1}$$

By construction, the following adding-up constraints must hold:

$$\sum_{i=1}^{N} a_{ij} = 1 \qquad\qquad j = 1, 2...., N. \tag{2}$$

The return on the assets of an arbitrary firm i is then a weighted average of the yield of the firm's own project and the yields of the projects of the other firms. Subtracting from this the value of the firm's liabilities we obtain the following expression for the net financial position of firm i:

$$\sum_{j=1}^{N} a_{ij} \widetilde{R}_j - M_i. \tag{3}$$

Hence we see that linkages are present in the asset side of the firms' balance sheet, since the value of the assets of firm i may be related to the value of the assets of other firms $j \neq i$. In contrast, no linkage is present on the liability side.

Moreover, the fact that the return on any given project is subject to shocks, while the return promised to a firm's creditors is deterministic, implies that the (ex post) value of the assets of a firm may turn out to be lower than the value of its liabilities (the expression in (3) may have negative sign), in which case the firm must *default*. Default entails two types of costs. First, there are the liquidations costs – for simplicity, we assume that these costs leave no resources available to make any payment to creditors at the time of default. In addition, there are additional costs deriving from the fact that a defaulting firm stops operating and hence loses any future earnings possibility. These opportunity costs are assumed to be substantial, so that the value of a firm is maximized when its probability of default at any point in time is minimized

There is a large set of investors, who are the source of the supply of funds to firms. Investors are risk neutral and require an expected gross rate of return equal to r in order to lend their funds. Since firms may default, in which case creditors receive a payment equal to zero, the nominal gross rate of return on the deposits to the firms must be greater or equal than r. Specifically, if we denote by φ_i the probability that any given firm i defaults (an endogenous variable), the gross return on the deposits of this firm must equal:

$$M_i = \frac{r}{1-\varphi_i}. \tag{4}$$

The presence of financial linkages among firms may generate contagion, that is, shocks hitting the project of one firm may propagate through the system, affecting the value of the assets of several firms, and generate widespread default.

REMARK 1 *In the environment considered here the mutual exposure between firms is due to the fact that firms are exposed to common shocks, given the cross-ownership of claims on the yields of the different projects. Hence the default of one firm has no direct implication for the solvency of other firms, and the possibility of contagion from a large shock hitting one firm only comes from the correlation of the returns among the assets of different firms.*[7] *The specifications in Acemoglu et al. (2013) as well as Glasserman and Young (2013) differs from ours in that*

[7] Lazear (2011) in a widely referenced Wall Stree Journal article, refers to this as the 'popcorn' mechanism, contrasting it to the 'domino' effect, where it is the default of one firm that triggers the contagion to other firms.

financial linkages among firms are given by mutual borrowing and lending relationships and thus concern both the asset and liability side of the firms' balance sheet. On the other hand, in Elliott et al. (2013) linkages only affect the asset side, as here, but are due to cross-ownership of equity instead of assets. In these three papers it is the default of a firm that produces (and, in the last one, amplifies, due to the fixed costs associated with bankruptcy) the contagion from a shock hitting the firm. Cabrales et al. (2014) show that the main difference is that in the aforementioned cases the net financial position of firms is a non-linear, possibly discontinuous function of the realization of the returns on the investment projects of the various firms in the system, while in our model it is a linear function - recall (3). And, more importantly, Cabrales et al. (2014) also show that the characterization of the pattern of contagion for different realizations of the size of the shocks hitting the system is similar in the four models.

Welfare Different specifications of the pattern of financial linkages among firms, as described by the matrix A, entail different levels of exposure of each firm to the shocks that may hit the system. When a shock of size L hits the return on the project run by some firm i, the exposure to it of any firm j in the system is given by $a_{ji}L$. It then follows from (3) that firm i defaults in response to such a shock when

$$a_{ii}(R-L)+\sum_{j\neq i}a_{ij}R<M_i, \quad \text{that is,} \quad a_{ii}L>R-M_i,$$

while any firm $j\neq i$ defaults whenever

$$\left(a_{jj}+\sum_{k\neq i,j}a_{jk}\right)R\,+\,a_{ji}(R-L)<M_j.$$

Hence the lower is a_{ii} the less firm i is exposed to the shocks hitting the return on its project, and the larger is the ability of the firm to withstand such shocks without defaulting. However, given (2), this also means that the larger is the exposure a_{ji} to that shock of other firms $j\neq i$, so that these firms will also be more affected when the project of firm i is hit.

Thus when a firm establishes financial linkages with other firms, for instance by exchanging assets with them, the firm reduces its exposure to the shocks hitting its own project but at the same time it becomes exposed to the shocks affecting the projects of those firms with which the firm in question is connected. In the environment considered there is thus a *trade-off between risk sharing and contagion*. Forming linkages allows firms to be more resilient to the shocks hitting their own projects, but it also exposes them to the shocks that may hit other firms in the system.[8]

One of the primary objectives of this paper is to compare the performance of different financial structures, that is different patterns of financial linkages, in terms of social welfare. Our assumptions that default costs are very high and that all firms are risk neutral and *ex ante* identical imply that social welfare is maximized when the expected number of defaults in the system is minimized.[9] Since investors are risk-neutral and they always earn an expected rate of return r, their welfare is always the same across the different configurations. It can be easily checked that this criterion is equivalent to that of minimizing the individual probability that any single firm defaults.[10] Thus, from an *ex ante* viewpoint, social and individual objectives are aligned.

To analyze how different financial structures perform in the light of this trade-off between risk sharing and contagion, it is useful to allow for sufficiently rich probability distributions of the shocks, where shocks of different size may occur. Here is where the linearity property of the effects of the shocks on the system, highlighted in Remark 1, comes useful as it allows to make the analysis tractable.

Probability Structure of the shocks We specify now some properties of the probability distribution of the shock $\widetilde{L}_i$ which may hit the yield of the project of firm i. Conditionally

[8]This rationale for forming financial linkages is not present in models where these linkages are given by borrowing and lending relationships among firms. In these models, default is always minimal in the absence of any linkage.

[9]Beale et al. (2011) propose a similar criterion to evaluate and compare different financial systems, based on the minimization of a "systemic cost function" defined as the expectation of a convex function of the number of defaults in the system. Thus, in this case, not only the expected number of defaults matters, but also its variability.

[10]To see this, note that the *ex ante* probability of default of an individual firm φ is equal to $\sum_m \rho(m)\varphi(m)$, where $\rho(m)$ stands for the probability that m firms default and $\varphi(m)$ for the conditional probability that any particular firm defaults when there are a total of m defaults in the system. Then, since

$$\varphi(m) = \frac{1}{N} + (1 - \frac{1}{N})\frac{1}{N-1} + \cdots + (1 - \frac{1}{N} - \cdots - \frac{1}{N})\frac{1}{N-m+1} = \frac{m}{N}$$

we obtain that the expected number of defaults in the system is $\sum_m \rho(m)\, m = \varphi N$, i.e. is proportional to the individual default probability.

on such shock hitting firm i, with probability π the shock is 'small' (labeled s) and the project experiences a loss of size $\underline{L}$. With probability $1-\pi$ the shock is 'big' (labeled b), and is described by a random variable $\widetilde{l}$, with support[11] $[\underline{L}, \infty)$ and cumulative distribution function $\Phi(l)$. Summarizing, the gross return on the project of an arbitrary firm i is:

$$\widetilde{R_i} = \begin{cases} R \text{ with prob. } 1-q \\ R-\underline{L} \text{ with prob. } q\,\pi \\ R-\widetilde{l} \text{ with prob. } q(1-\pi) \end{cases} \tag{5}$$

For the risk of default to be an issue, we assume:

A1. (i) $R\,(1-q) > r$

(ii) $R - \underline{L} < r$.

The first inequality ensures that a firm's project is viable, that is, its expected return exceeds what must be paid to lenders. On the other hand, the second inequality implies (since $r \leq M$ and $\widetilde{l} \geq \underline{L}$) that if a firm can only draw on the revenue generated by its project, it is surely unable to pay depositors (and hence must default) whenever a shock, *whether small or large* hits its return.

We assume that shocks are rare and thus each period at most one firm is hit by a shock. Given (5), this can be motivated by postulating that, even if shocks hit firms in a stochastically independent manner, the probability q that a shock hits any given firm is so low that the probability that two or more shocks arrive in a single period is of an order of magnitude that can be ignored.[12]

On the nature of those shocks, we make the following key assumption:

A2. (i) $\pi > N(1-\pi)$.

(ii) $\frac{1}{2}(R-\underline{L}) + \frac{1}{2}R \geq \frac{r}{1-Nq(1-\pi)}$.

[11] We allow the support of $\widetilde{l}$ to go to ∞ purely for technical convenience. The analysis could have been equivalently carried out by truncating the distribution at some upper bound $\bar{L}$ smaller than R so as not to violate limited liability, but with R, $\bar{L}$, r big enough.

[12] Or, as an extreme formalization of this idea, we could model time continuously and assume that the arrivals of small and big shocks to each firm are governed by independent Poisson processes with fixed rates π and $1-\pi$, respectively. Then, the probability that two shocks arrive simultaneously is zero.

Part (i) of the above assumption says that s (small) shocks are significantly more likely than b (big) ones. In particular, it is more likely that a given firm is hit directly by a s shock than a b shock hits any of the firms in the system. Part (ii) then ensures that the presence of sufficient linkages to other firms allow a firm to fully insure against the s shocks, i.e. guarantees that it never defaults when an s shock hits a firm in the system. To understand the statement of the condition, note that the term on the right-hand side of A2(ii) constitutes an upper bound on the gross rate of return M that must be paid to investors by a firm that is always solvent when an s shock hits anywhere in the system. The probability of default φ of such a firm is in fact not larger than $Nq(1-\pi)$, which is the probability that a b shock hits one of the N projects. The term on the left-hand side of the inequality in A2(ii) is then a lower bound on the gross revenue generated by the assets of a firm that owns a fraction not greater than one half of the total outstanding amount of claims to the return on its own - or on any other specific firm's - project,[13] and the complement fraction of claims to projects of other firms.

2.2 Financial Structures

The set of possible financial structures, or networks, as described by the matrix A, is potentially quite large. The elements of A specify not only which firm has a financial linkage with any given firm i, but also the intensity of this linkage. It is then useful to identify three key elements that characterize possible patterns of these linkages.

The first element is the degree of *externalization of risks*, i.e. the strength of linkages to other firms, as captured by the relative value of the diagonal terms of A with respect to the sum of all their corresponding off diagonal terms. Clearly with no financial linkages, that is when $A = I$, no contagion is possible, but every firm is in isolation and has to bear the whole shock hitting the returns on its project, i.e. there is no risk sharing.

The second element is the degree of *segmentation* that describes the extent to which a firm is linked to all or a subset of the other firms. With no segmentation, every firm is linked to every other firm, while with maximal segmentation the system of N firms is divided into disjoint components of minimal size (one or two according to whether the

[13] That is, $a_{ij} \leq 1/2$ for all i, j.

degree of externalization is zero or positive).

The third element is the degree of *network density*, that is the relative strength of the connections to the firms to which a firm is linked. When the density is high the intensity of the linkages of any firm i, as measured by the value of a_{ij}, displays similar values for all firms j to which i is linked to. Hence firm i has similar exposure to all the firms it is linked to. On the other hand, when the density is low, the exposures a_{ij} vary considerably with j. As we will see we can interpret these two cases as describing, respectively, a situation where the network distance between any two firms in a component is small and another where a firm is directly linked to few firms and indirectly linked, with varying distance, with most other firms.

Assumptions A1(ii) and A2(i) imply that a firm with no linkages always defaults when hit by an s shock and that s shocks are much more likely than b shocks. It then follows from A2(ii) that the probability of default of a firm is always lower when its degree of externalization is not too small (at least equal to one half) than when it is in isolation. That is, some degree of risk sharing is beneficial as it allows to reduce expected default. Given this property we will consider network structures that exhibit the same degree of externalization, normalized to be equal to 1/2, that is, for all i we have $a_{ii} = 0.5$ and $\sum_{j \neq i} a_{ij} = 0.5$.[14] Financial structures will always exhibit the same level of risk externalization with other firms and only differ for how the 1/2 fraction of the externalized risk is distributed among the rest of the firms. Hence the focus will be on financial structures that differ in terms of segmentation and density, which determine whether this risk is distributed among many or few firms, and whether or not it is equally distributed among them.

While *any* financial structure exhibiting this sufficient degree of externalization allows firms to attain full insurance against s shocks, their performance in terms of default probability when b shocks hit the system may be markedly different. The case of maximal integration among firms is the one of no segmentation and high density, where each firm has a financial linkage, and of the same intensity, with any other firm in the system. This

[14]The presence of an upper limit on the degree of externalization can be justified by considerations of moral hazard. This requirement is analogous, for example, to a well-known provision in the recent Dodd-Frank act passed in the USA to strengthen the regulation of the financial system. By virtue of this new legislation, under certain circumstances "a securitizer is required to retain not less than 5 percent of the credit risk..." (see http://www.sec.gov/about/laws/wallstreetreform-cpa.pdf).

is the case when the range of contagion is the widest. Segmentation and lower density of connections are two different ways of limiting the range of contagion. One of the primary aims of this paper is indeed to understand which configurations limit contagion best given the properties of the probability distribution of the b shocks.

In addition, given that, for the moment, all firms are assumed ex-ante symmetric it is natural to focus our attention on the case where the pattern of linkages within each component is also symmetric across firms. This implies that A is symmetric, i.e.

$$a_{ij} = a_{ji} \quad \text{for all} \quad i, j = 1, 2, ..., N. \tag{6}$$

We will allow however for asymmetries in the sizes of the various components.

We describe now more in detail the set of financial networks that we consider and how they differ along the two other dimensions described above.

Segmentation We will allow for any possible partition of the N firms into disjoint components. Each component is formed by firms among which a financial linkage exists, while there is no linkage among firms lying in different components. It is convenient to denote by K_l the number of firms to which a firm lying in component l is linked to; the size of the component is then $K_l + 1$. In light of the above, K_l can range from 1 to $N - 1$. In terms of the matrix A, rows and columns can always be rearranged so that this matrix has a block diagonal structure.

When all components have the same size, a simple measure of the degree of segmentation in the system is given by the number $K + 1$ of firms in each component, or equivalently by the number $C = N/(K + 1)$ of equal-sized components in which the set of firms is divided. The larger the segmentation, the lower the number K of firms indirectly affected by a given shock but, *ceteris paribus*, the larger their mutual exposure and hence the probability of default if a b shock hits them. At the two extremes of segmentation, we have the case $K = N - 1$, where all firms are connected (directly or indirectly), and $K = 1$, where each firm only engages in trade with a single other firm. We shall allow for all possible values of K between 1 and $N - 1$.

Network Density In this regard we shall consider two cases. The first one where the intensity of the linkage between any pair of firms in a component is the same. Hence the pattern of connections in a component of size $K+1$ is described by a square matrix A_K of the form:

$$A_K = \begin{pmatrix} 1/2 & 1/2K & \cdots & 1/2K \\ 1/2K & 1/2 & \cdots & 1/2K \\ \vdots & \vdots & \ddots & \vdots \\ 1/2K & 1/2K & \cdots & 1/2 \end{pmatrix}, \tag{7}$$

In this case each firm is equally exposed to the shocks hitting any other firm in its component.

In the second case the pattern of exposure of a firm i takes a different value for all j. More specifically, after a suitable rearrangement of rows and columns of A_K we posit without loss of generality that the following property holds[15]:

$$a_{ij} > a_{lq} \text{ when } |i-j| < |l-q|. \tag{8}$$

To get some understanding for the possible mechanisms that can underlie these patterns of financial linkages among firms we illustrate in the example below how these linkages can be obtained as a result of exchanges among firms, possibly repeated, of claims to the returns on the assets in their balance sheet. We can then use a network to specify, for any given firm, which other firms are its direct neighbors, that is firms with which a direct exchange of assets can take place. The pattern of linkages with high density, as in (7) above, is obtained when each firm is directly linked with any other firm (in a component), that is, the network of trades is a *complete* network. The pattern with lower density, (8), is instead obtained when each firm is directly linked to two other firms, and only indirectly linked, with varying distance, with the other firms: the network of trades can be intuitively conceived in this

[15] Given our assumption that $a_{ii} = 0.5$ for all i, the pattern of exposure is exactly the same for the complete and the ring structure if the component size is minimal (i.e. $K = 1$). The difference between the two structures then grows wider the higher is K.

case as a *ring* network. In the rest of the paper we will thus refer to these patterns of linkages as, respectively, a complete and a ring network. It should be clear, however, that what we label as a ring network refers more generally to situations where the exposure of a firm towards other firms in its component varies (and can be obtained as result of patterns of trades in *intermediate network structures* between the complete and the ring).

EXAMPLE 1 *[Exchanges of assets in a network]*

Consider a set of $K+1$ firms, arranged in an (undirected) network, either a complete network or a ring. Each firm, initially endowed with the claims to the yield of its own project, exchanges a fraction $1-\theta$ of its assets, for the same fraction of assets held by other firms. This swap of assets is then repeated, but after the first round already concerns shares in the composite portfolio of assets acquired through the previous trades. We allow these trades to be iterated a certain number m of times. In this way a firm ends up holding claims to the returns on the projects of firms with whom it is only indirectly linked with (that is, there is a multiple-link path in the trade network connecting the two firms). These exchanges of assets can be viewed as reflecting a process of repeated rounds of securitization and trade of the assets of a financial firm in order to diversify its risks.

In the case of a complete network, the pattern of asset exchanges is described by

$$B_K^c = \begin{pmatrix} \theta & (1-\theta)/K & (1-\theta)/K & \cdots & (1-\theta)/K & (1-\theta)/K \\ (1-\theta)/K & \theta & (1-\theta)/K & \cdots & (1-\theta)/K & (1-\theta)/K \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (1-\theta)/K & (1-\theta)/K & (1-\theta)/K & \cdots & (1-\theta)/K & \theta \end{pmatrix}, \tag{9}$$

while in the case of a symmetric ring structure it is

$$B_K^r = \begin{pmatrix} \theta & (1-\theta)/2 & 0 & \cdots & 0 & (1-\theta)/2 \\ (1-\theta)/2 & \theta & (1-\theta)/2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (1-\theta)/2 & 0 & 0 & \cdots & (1-\theta)/2 & \theta \end{pmatrix}. \tag{10}$$

The pattern of financial linkages obtained by iterating m times these trades is then, respec-

tively, $(B_K^c)^m$ and $(B_K^r)^m$. It is immediate to see that $(B_K^c)^m$ has the same uniformity in the intensity of linkages to other firms as (7), while $(B_K^r)^m$ satisfies the same intensity-decaying condition with network distance as (8), provided $m \geq K/2$.

2.2.1 The Continuum Approximation

The pattern of risk exposure induced by the matrix A can be graphically depicted through a function that, for each firm i, specifies the fraction of the claims to the yield of the project of firm i that is held by any other firm j in the component. This function proves particularly convenient to determine the expected probability of default of a firm. Its values vary with the network distance of i to other firms j and are given by the terms on the corresponding row of the matrix A. Since $a_{ii} = 1/2$ the "exposure function" always reaches the highest value when $i = j$. In the particular case of the complete structure (as in (7)) it takes a constant value for all other $j \neq i$, while for the ring (as in (8)) it is monotonically decreasing step function.

Because of the discreteness of the domain of this function a formal analysis of the firms' probability of default for different specifications of the probability distributions of the b shocks becomes quite involved and tedious to carry out. To render the analysis more tractable, we study in what follows a continuum approximation of our model that allows to approximate the discrete exposure function by a continuous real function that embodies the same features as the original one.[16]

In the continuum approximation of the model, N is taken to be the *measure* of firms in the system, while $K + 1$ is the generic notation used to represent the measure of firms belonging to a certain component. The returns on a firm's project are the same as in (5). In correspondence with the discrete formulation, when a shock occurs it is taken to hit directly a unit mass of firms in one component. This mass of firms, therefore, plays the role of the single firm directly hit by a shock in the discrete context.

Segmentation is modeled as in the previous section, except that the size of a component,

[16]This continuum formulation can be seen as representing a limit description of a context consisting of a large number of firms, each of them of small (relative) size. This interpretation, however, is not essential, so we prefer to view it simply as a continuous approximation of a context where the number of firms is not necessarily large.

$K+1$, need not be an integer and can now be any real positive number lying between 1 and N. Differences in network density, on the other hand, are modeled as follows.

In the case of financial structures with complete components, the pattern of exposure to the returns on other firms' projects is as in the discrete model: the exposure to a shock that hits any other firm in the same component is constant and equal to $1/2K$, while it is zero to firms in other components. On the other hand, if components have a ring structure (density is low), the pattern of exposure is described by a continuous function $f(d;K)$, where $d \in [0, K/2]$ is the ring distance to the set of firms directly hit by the shock. For the sake of tractability we posit that function has the following form:

$$
\begin{aligned}
f(d;K) &= \tfrac{1}{2} - \tfrac{K-1}{2}d && \text{for } 0 \le d \le 1/(K+1) \\
&= \tfrac{K}{K(K+1)-2} - \tfrac{2}{K(K+1)-2}d && \text{for } 1/(K+1) < d < K/2 \\
&= 0 && \text{for } d = K/2.
\end{aligned} \tag{11}
$$

The function is defined for all $K \geq 1$ and it is immediate to verify that it exhibits the following properties that match those of the exposure function in the discrete set-up:

$$f(d;K) \text{ is positive and decreasing for all } K/2 > d > 0, \tag{12}$$

$$f(0) = \frac{1}{2}, \tag{13}$$

$$f(K/2) = 0, \tag{14}$$

$$2\int_0^{K/2} f(x;K)\mathrm{d}x = \frac{1}{2}. \tag{15}$$

First, (12) says that every firm in a component is affected by a shock hitting any other firm in the component, but with an intensity that decreases with the distance to the source of the shock. In addition, (13) states that the level of exposure to a firm at minimal distance is the same as that to a direct shock, while (14) says that the exposure becomes vanishing small when the distance to the source of the shock is maximal in the component. Finally, (15) is the adding-up constraint which reflects the requirement that firms externalize the fraction 1/2 of their risk. Conditions (12), (13), (14) and (15) capture the essential features displayed by the exposure function in the discrete setup.

Note that $f(d;K)$ is a two-piece linear function with the kink at the bisectrix (i.e. at a distance $d = 1/(K+1)$ such that $f(1/(K+1)) = 1/(K+1)$). An illustration of how it approximates the original function for the discrete setup is displayed in Figure 1.

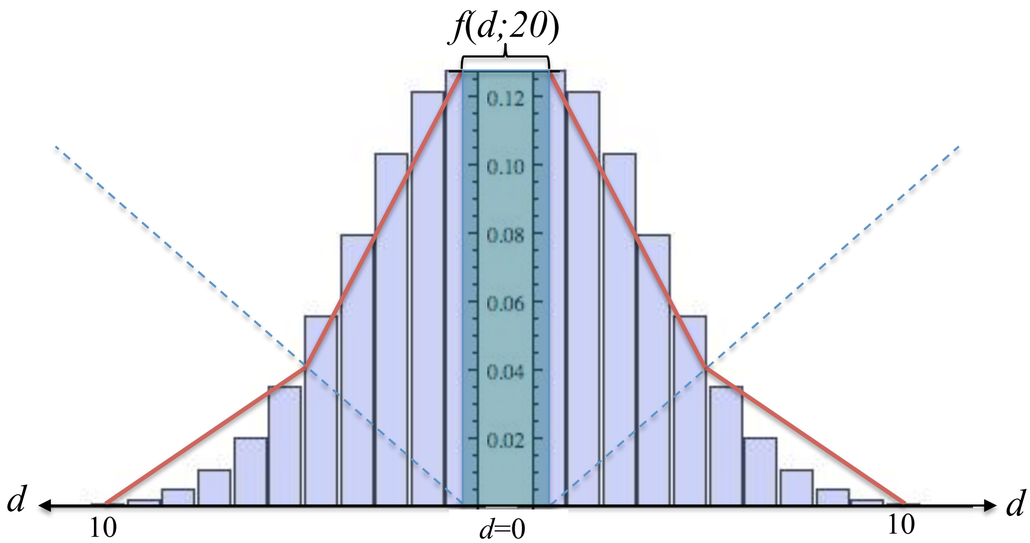

Figure 1: Continuum approximation of the exposure function. The step function describes the exposure in a ring with $K+1 = 21$ firms, as a function of the distance d to either side of the location occupied by any given firm. The exposure is the one induced by the matrix $A_{20} \equiv (B_{20}^r)^{10}$, with B_{20}^r being the matrix of asset exchange specified in (10). In contrast, the solid line describes the exposure for the continuum approximation $f(d, 10)$.

3 Optimal Financial Structures

In this section we compare the welfare properties of alternative financial structures, that differ in terms of the degree of segmentation and network density. As we explained, in order to assess welfare, we need to compute the probability of bankruptcy of firms in the system. Let $\varphi_i(A; \Phi(\cdot))$ denote this probability for an arbitrary firm i when the financial network structure is described by the matrix A and the distribution of the b shocks is given by the cumulative distribution function $\Phi(\cdot)$. It satisfies the following:

$$\varphi_i(A; \Phi(\cdot)) = q\pi_b \Pr\left\{0.5\left(R - \widetilde{l}\right) + 0.5R) < M_i\right\} + q\pi_b \sum_{j \neq i}^{N} \Pr\left\{R - M_i < a_{ij}\widetilde{l}\right\} \quad (16)$$

where M_i fulfills (4). Given the assumed symmetry (6) of each component, M_i takes the same value for all firms in a component. The first term on the right hand side is the probability of default when a b shock hits firm i, the second term is the probability of default of the same firm when a b shock hits any of the other $N-1$ firms. As argued before,[17] the expected number of defaults in the system is $\sum_{i=1}^{N} \varphi_i (A; \Phi(\cdot))$.

It is worth highlighting that the first term on the right hand side of (16), giving the probability of default of a firm whose project is hit by a b shock, is the same across all structures considered. These different structures may only vary in their ability to limit contagion – that is, in how effective they are in preventing large b shocks from inducing the default of firms connected to the one directly hit by the shock. For this comparison to be non trivial, we assume that the number of firms in the system is not too small, that is:

A3. $N > 4$

In what follows we will determine the expected mass of firms in the system that default for the continuum approximation of the model. Note first that the unit mass of adjacent firms directly hit by a shock will default if, and only if[18],

$$l > (R - M)\, 2.$$

If this inequality does not hold, the firms hit by the shock do not default and no other firm in the corresponding component defaults either. This simply follows from the fact that $1/2 \geq 1/(2K)$ and $f(d; K) \leq 1/2$ for any $d \geq 0$. But if those firms directly affected by the shock do default, what happens to all the others in the component naturally depends on the size $K+1$ of the component and on the pattern of the connections within it.

In the case where firms are connected through a *complete* network, the uniformity of the exposure among them has the following immediate implication: *all* firms indirectly affected

[17] See footnote 10.

[18] As argued above, this condition is the same for all the financial structures considered. Also, by the assumed symmetry of linkages within each component, M_i will have the same value for all firms in a component, hence we will ignore the subscript i unless when needed.

(i.e. not hit by the shock but lying in the component affected) will default if

$$l > 2K\,(R - M) \tag{17}$$

whereas *none* of those firms will default otherwise. Thus, if we let $g_c(l; K, M)$ stand for the mass of firms that default when a shock of size l hits some other firm in their component, with the gross return on their liabilities being M, that magnitude is given by the following step function:

$$g_c(l; K, M) = \begin{cases} 0 & \text{if } l \leq 2K\,(R - M) \\ K & \text{if } l > 2K\,(R - M) \end{cases} \tag{18}$$

In contrast, when the firms in the component are connected through a *ring* network (as captured by the exposure pattern $f(\cdot)$), the conclusion is, in general, not so dichotomic. For, in this case, whether any particular firm in the component defaults or not depends on its ring distance d to those firms that have been directly affected. Such a firm defaults if, and only if,

$$l > \frac{1}{f(d; K)}\,(R - M)\,,$$

which is to be contrasted with (17). Hence the threshold that marks the relevant "default range" is given by the distance $\hat{d}$ such that

$$f(\hat{d}; K) = \frac{R - M}{l}, \tag{19}$$

so that a firm defaults if, and only if, its distance d from the set of firms directly hit by the shock satisfies $d < \hat{d}$. In a ring structure, therefore, the mass of firms defaulting is not a discontinuous function of the size l of the shock as in the complete structure. Rather, as l increases, the mass of firms defaulting among those indirectly affected by it grows gradually (see Figure 2 for an illustration), as determined by the function $g_r(l; K, M) \equiv$

$2f^{-1}((R-M)/l;K)$ given by:

$$g_r(l;K,M)=\begin{cases}0 & \text{for } l\le 2(R-M)\\ \frac{2}{K-1}-\frac{4}{K-1}\frac{R-M}{l} & \text{for } 2(R-M)\le l\le (R-M)(K+1)\\ K-[K(K+1)-2]\frac{R-M}{l} & \text{for } l\ge (R-M)(K+1)\end{cases} \tag{20}$$

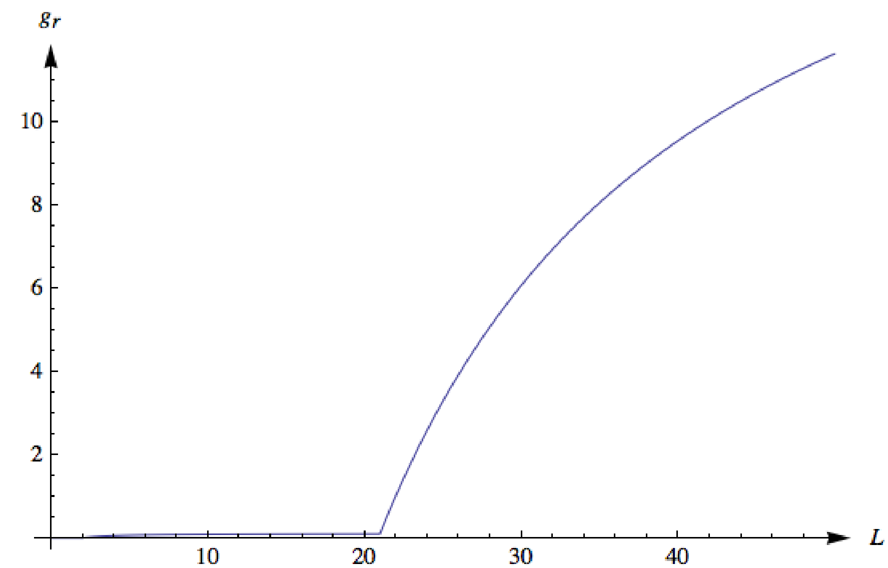

Figure 2: The function $g_r(l;20,M)$, specifying the mass of firms that default when indirectly hit in a component of size $K+1=21$ for any given magnitude l of the b shock, plotted for the normalization $R-M=1$, as posited below.

We rely on the functions given in (20), and (18) to determine the optimal pattern of segmentation and best network density. By segmentation we mean the number C and hence the sizes K_i+1 of the separate components i, $i=1,..,C$. On the other hand, by network density, we refer to the choice between a ring or a completely connected structure within each component. Specifically the objective is to minimize the expected mass of firms defaulting in the system, given by

$$\varphi_\nu = qN\pi_b\left(\frac{1}{N}\Pi+\sum_{i=1}^{C}\frac{K_i+1}{N}\frac{K_i}{N}\frac{\mathbb{E}g_\nu(\widetilde{l};K_i,M_i)}{K_i}\right) \tag{21}$$

where the subindex $\nu=c$, r signifies that either a complete or a ring structure is considered, and Π is the probability that a firm defaults when directly hit by a b shock. The above expression reflects the fact that we allow for the possibility of asymmetric structures, with components of possibly different size, in which case the ex-ante probability that any firm defaults is the average weighted probability that firms in each component default.

Consider first a fixed structure $\nu \in \{c, r\}$. Then, discarding irrelevant terms in (21), the optimal degree of segmentation is obtained as a solution of the following optimization problem:

$$\begin{array}{c} \min_{K_i, C} \sum_{i=1}^{C} \frac{K_i+1}{N} \mathbb{E} g_\nu(\widetilde{l}; K_i, M_i) \\ \text{s.t.} \sum_{i=1}^{C} \frac{K_i+1}{N} = 1 \end{array} \tag{22}$$

We will show that the solution is (essentially) symmetric for all cases under consideration, with $K_i = K$ for all i. This implies that problem (22) can be reduced to minimizing $\mathbb{E} g_\nu(\widetilde{l}; K, M)$ with respect to K. Then, by comparing the optimum values of the solution of (22) for $\nu = r$ and $\nu = c$ we obtain the optimal network density.

Note that the value of $\mathbb{E} g_\nu(\widetilde{l}; K, M)$ depends not only on the distribution $\Phi(l)$ but also on the other parameters of the model, $R, r, \underline{L}$. The specific values of these parameters have however little interesting bearing on the analysis and our primary focus is on the effects of the shock distribution on the optimal network structure. Hence in what follows we normalize $R - r$ to unity and set also $\underline{L} = 1$. Furthermore, even though M is an endogenous variable (determined by (4)) and varies with the underlying network structure, for the purpose of determining the optimal structure we can make it equal to r (and hence $R - M = 1$) if $qN\pi_b$ is small. More formally, we can show that:

Lemma 1 $\lim_{qN\pi_b \to 0} M = r$.

Therefore, in identifying the optimal structure, we can set $M = M_F(0) = r$ and thus $R - M = 1$.

We organize our formal analysis in the rest of this section in three parts. First, in Subsection 3.1 we identify some clear-cut conditions on the probability distribution of the b shocks under which the optimal segmentation is one of the two polar extremes – maximal or minimal – and the optimal degree of connectivity is complete. Then, in Subsection 3.2 we extend the analysis to more general specifications of the shock distribution, for which intermediate levels of segmentation are optimal. Finally, in Subsection 3.3 we identify scenarios where the optimal structure exhibits not only intermediate levels of segmentation but also low density of connections, as embodied by the ring structure.

3.1 Polarized segmentation

In order to get a clear understanding of the forces at work, we shall start by examining the case where the probability distribution of the b shocks is of the Pareto family with support $[1, \infty)$ and density $\gamma/l^{\gamma+1}$. By modulating the decay parameter γ, this formulation already allows the discussion of many questions of interest such as the contrast between fat and thin tails in the shock distribution (i.e. between scenarios where large shocks are relatively frequent or not). As mentioned above, our analysis will be carried out in two steps. Firstly, we shall characterize how γ affects the optimal degree of segmentation (as described by K) for the ring and for the complete structure. Secondly, we shall compare these two structures.

Let $D_r(K,\gamma) = \mathbb{E}_\gamma g_r(\widetilde{l}; K)$ denote the expected mass of firms in a ring of size $K+1$ who default when indirectly hit by a b shock with a Pareto distribution with parameter γ (that is, when the shock hits some other firm in the component).[19] We have:

$$\begin{aligned} D_r(K,\gamma) &= \int_1^\infty \ g_r(l;K)\ \mathrm{d}\Phi(l;\gamma) \\ = \int_{K+1}^\infty \ \left(K - [K(K+1)-2]\tfrac{1}{l}\ \right)\tfrac{\gamma}{l^{\gamma+1}}\mathrm{d}l &+ \int_2^{K+1}\left(\tfrac{2}{K-1} - \tfrac{4}{K-1}\tfrac{1}{l}\right)\tfrac{\gamma}{l^{\gamma+1}}\mathrm{d}l \\ = \gamma\left[K\tfrac{1}{\gamma(K+1)^\gamma} - [K(K+1)-2]\tfrac{1}{(\gamma+1)(K+1)^{\gamma+1}}\right] &+ \\ 2\gamma\left[-\tfrac{1}{K-1}\tfrac{1}{\gamma(K+1)^\gamma} + \tfrac{2}{K-1}\tfrac{1}{(\gamma+1)(K+1)^{\gamma+1}} + \tfrac{1}{K-1}\tfrac{1}{\gamma\, 2^\gamma} - \tfrac{2}{K-1}\tfrac{1}{(\gamma+1)2^{\gamma+1}}\right]& \end{aligned} \tag{23}$$

We study next how the above expression behaves as K varies in its admissible range (recall that the minimum[20] admissible value of K is 1 and its maximal value is $N-1$). We establish in the following proposition that whenever $\gamma > 1$ (i.e. the distribution function of $\widetilde{l}$ does not have fat tails) $D_r(K,\gamma)$ attains a minimum at the highest value of K, i.e. $N-1$. On the other hand, when $\gamma < 1$ (i.e. the distribution has fat tails) the function attains a minimum at $K = 1$.

[19] As discussed at the end of the previous section, we shall focus our attention on the case where $M = r = R-1$ and omit then this argument from the functions g_v throughout.

[20] Strictly speaking, the expression in (23) holds for $K > 1$. For $K = 1$ we have $D_r(1,\gamma) = \gamma\left[K\frac{1}{\gamma(K+1)^\gamma} - [K(K+1)-2]\frac{1}{(\gamma+1)(K+1)^{\gamma+1}}\right]$. However, since we show in the proof of Lemma 2 that $D_r(K,\gamma)$ is continuous at $K=1$, in what follows it suffices to work with (23).

Lemma 2 *When the shock $\widetilde{l}$ has a Pareto distribution, the component size which minimizes the expected mass of firms defaulting for the ring structure is minimal ($K = 1$) if $\gamma > 1$, and maximal ($K = N - 1$) if $\gamma < 1$.*

On this basis, we can easily determine the optimal segmentation pattern in the system. To this end we must also take into account the constraint present in (22), $\sum_{i=1}^{C}(K_i+1)/N = 1$. Given the findings of the above lemma, this constraint only binds when N is odd and $\gamma < 1$, as in such case a structure with all components of the optimal (minimal) size 2 is not feasible. To find the optimal structure in this case we establish the concavity of $D_r(K, \gamma)$:

Lemma 3 *When $\gamma < 1$ the function $D_r(K, \gamma)$ is strictly concave in K, for all $K > 1$.*

From Lemmas 2 and 3 it immediately follows that the optimal ring structure when $\gamma < 1$ and N is odd is "almost" symmetric, with $\frac{N}{2} - 1$ components of minimal size 2 and a residual component of size 3. In all other cases, the optimal ring structure is the one with all components of the same optimal size, determined in Lemma 2. We can then summarize our findings in the following statement:

Proposition 1 *Suppose the shock $\widetilde{l}$ has a Pareto distribution. With $\gamma > 1$ (no fat tails) the optimal degree of segmentation for the ring structure is minimal, with a single component including all N firms. Otherwise, with $\gamma < 1$ segmentation is maximal, with all components of the minimal size 2 (except one, of size 3, when N is odd).*

The previous result shows that there is indeed a trade-off between risk sharing and contagion. On the one hand, when the distribution of the shocks exhibits fat tails (hence large shocks are relatively likely), the predominant consideration is to control contagion rather than achieve risk sharing. Hence the expected number of defaults is minimized by breaking the network into disjoint components of minimal size, which limits the extent to which a shock may spread its consequences far into the system. Instead, when the distribution has no fat tails, the most important consideration becomes risk-sharing, which is maximized by placing all firms in a single component.

Next, we turn to studying the analogous question for the case where the components are completely connected. In this case, the expected mass of firms who default in a completely connected component of size $K+1$ when indirectly hit by a b shock is:

$$D_c(K,\gamma) = \mathbb{E}_\gamma g_c(\widetilde{l};K) = K \ \Pr\left(\widetilde{l} \geq 2K\right) = K \ \left(\frac{1}{2K}\right)^\gamma . \tag{24}$$

Hence

$$\frac{\partial D_c}{\partial K} = -\left(\gamma-1\right)\left(\frac{1}{2K}\right)^\gamma \gtrless 0 \quad \Longleftrightarrow \quad \gamma \lessgtr 1,$$

which readily implies that the optimal component size is again minimal when $\gamma < 1$ (the shock distribution has fat tails), while it is maximal in the case where $\gamma > 1$. Hence the optimal segmentation structure is the same as for the ring, with all components of the optimal size when the constraint $\sum_{i=1}^{C}(K_i+1)/N = 1$ does not bind. In contrast, when $\gamma < 1$ and N odd, and hence the former constraint binds, the optimal structure is different from the one obtained for the ring in this situation. It is now exactly symmetric, with $\frac{N-1}{2}$ components, all of the same size (slightly larger than 2). This is because, as can be seen from the proof of the Proposition 4, $D_c(\cdot)$ is a convex function of K (remember that $D_r(\cdot)$ is a concave function of K). Hence the following result holds:

Proposition 2 *When the shock $\widetilde{l}$ has a Pareto distribution, the optimal degree of segmentation for the completely connected structure is minimal (one single component) if $\gamma > 1$, and maximal, with $N/2$ ($(N-1)/2$ if N is odd) identical components, if $\gamma < 1$.*

Finally, we need to compare the optimally segmented ring and complete network structures in order to identify which of the two is optimal when not only segmentation but also network density can be chosen. In view of Propositions 1 and 2, it is enough to compare the expected mass of firms defaulting under either maximal or minimum segmentation for the ring and the complete structures when, respectively, γ is lower or higher than one. As already noticed, when K is at its minimal admissible value (1), the pattern of exposure is the same for the two structures, $g_c(l;1) = g_r(l;1)$ for all l, hence a difference only arises when the optimal component size is greater than 1 for at least one structure. This leads to the following result.

PROPOSITION 3 *If the shock $\widetilde{l}$ has a Pareto distribution, for all values of $\gamma < 1$, when N is even the optimal structure is equivalently completely connected or a ring structure, while for all $\gamma > 1$ and for N not too small ($N > 1 + (1+\gamma)^{\frac{1}{\gamma-1}}$) the complete network strictly dominates the ring structure.*

Combining the results obtained in this subsection, we conclude that when the distribution of the shocks has a simple Pareto structure and thus it either has, or does not have, fat tails, the optimal network always displays maximal density and polarized (maximum or minimum) segmentation. In this case, therefore, the whole adjustment to the underlying risk conditions (i.e. to the different values of γ) is obtained only by varying the segmentation pattern. However, as our subsequent analysis will show, neither the optimality of the polarized segmentation pattern nor of the complete connectivity within components are features maintained for other, more complex, shock distributions.

3.2 Intermediate Degrees of Segmentation

We show next that polarized segmentation is no longer optimal when the distribution of the shocks is more complex, as for instance when it is given by the mixture of two Pareto distributions.

PROPOSITION 4 *Suppose that the shock $\widetilde{l}$ is distributed as a mixture of a Pareto distribution with parameter $\gamma > 1$ and another Pareto distribution with parameter $\gamma' < 1$, with respective weights p and $1-p$. Then, there exist p_0, p_1, $0 < p_0 < p_1 < 1$, such that, whenever $p \in (p_0, p_1)$, the optimal pattern of segmentation for the completely connected structure is symmetric with components of intermediate size $K^* + 1$, $1 < K^* < N - 1$.*

Abusing slightly previous notation, denote by $D_c(K, \gamma, \gamma', p) = p\mathbb{E}_{\gamma} g_c(\widetilde{l}; K) + (1 - p)\mathbb{E}_{\gamma'} g_c(\widetilde{l}; K)$ the expected mass of firms who default in a complete component of size $K+1$ when indirectly hit by a b shock hitting a firm in the component, and the distribution of this shock is as in the statement of the above proposition. We show in the proof of this result that the function $D_c(K, \gamma, \gamma', p)$ attains a minimum at an intermediate value $\hat{K} \in (1, N-1)$. A symmetric structure with all components of size $\hat{K}$ is however generically not feasible now (i.e. violates the condition $\sum_{i=1}^{C}(K_i+1)/N = 1$). We then show that the optimal structure is still symmetric, with all components of size K^*, smaller or equal to $\hat{K}$.

A numerical analysis of the problem shows that a similar conclusion applies to the case where the components display a ring structure. That is, an intermediate level of segmentation continues to be optimal when the shock distribution involves a mixture of Pareto distributions with both fat and thin tails. However, since an analytic result in this case is hard to obtain, we illustrate matters through the following example.

EXAMPLE 2 *Set $\gamma = 2$, $\gamma' = 0.5$ and $p = 0.95$. For these values we find that the value of K which minimizes $D_c(K, \gamma, \gamma', p)$ is $\hat{K}^c = 5.65$, and at this value the expected mass of defaults (when a b shock hits some other firms in the component) is 0.13. The value of K that minimizes the corresponding expression for the ring structure, $D_r(K, \gamma, \gamma', p)$, is higher, $\hat{K}^r = 8.02$, and the expected mass of defaults in this case is also higher, equal to 0.145. The fact that $\hat{K}^r > \hat{K}^c$ can be intuitively understood as a reflection of the fact that, when arranged optimally, components with a ring structure compensate for their lower density of connections with a larger size. See Figure 3 for a graphical depiction of these two functions.*

To find the optimal financial structure for the whole system we also need to specify the value of N. Let $N = 10$. In this case it is clear that a symmetric structure with equal components of size $\hat{K}^c+1$ is not *feasible when the components are completely connected, and the same applies to components of size $\hat{K}^r + 1$ when they are rings. We find that both for the complete and the ring structures the optimal configuration of the system is given by two equal-sized components of size $K^* + 1 = 5$. Moreover, in line with the results stated above for the values of the mass of expected defaults at $\hat{K}^r$ and $\hat{K}^c$, we find that the optimal complete structure still dominates the optimal ring structure: the former yields an expected mass of defaults equal to 0.13 in contrast with 0.16 induced by the latter (see Figure 4). This conclusion is robust to alternative specifications of the parameter values of the environment.*

3.3 Sparse Connections

Let us consider now the case where the probability distribution of the b shocks is not smooth because it has some atoms. More precisely, let $\Phi(l)$ be the mixture of a Pareto

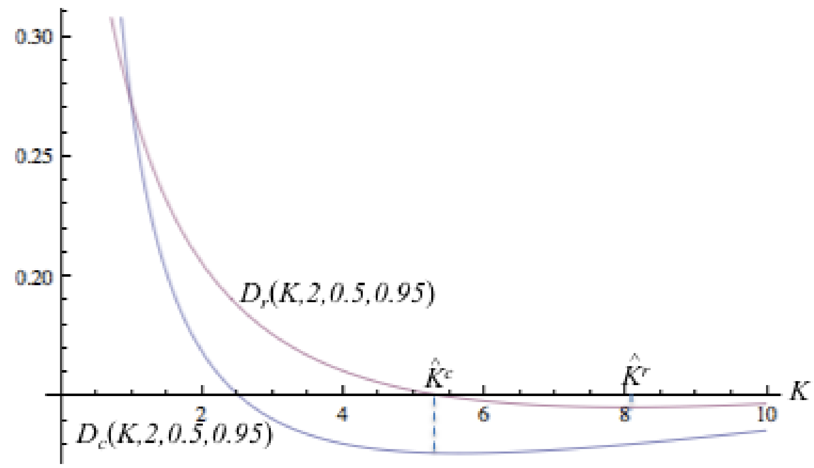

Figure 3: Expected mass of firms in a component of size K indirectly affected by a b shock who default when this shock hits their component, as given by the function $D_v(K, \cdot)$ of the component's size K, for both complete and ring structures ($v = c, r$). The parameter values are $\gamma = 2$, $\gamma' = 0.5$, $p = 0.95$.

distribution with $\gamma > 1$ and a Dirac distribution putting all probability mass on a shock of magnitude $\bar{L} > 2(N-1)$. On the one hand, note that the Pareto distribution considered has no fat tails. With such a distribution, as we saw in Section 3.1, minimal segmentation ($K = N - 1$) is optimal and, in addition, the completely connected structure dominates the ring structure. On the other hand, the shock $\bar{L}$ selected by the Dirac distribution has the following property. If a shock of that magnitude occurs and firms are arranged in a single component, all firms default when they are completely connected, while some survive if arranged in a ring. Combining the previous considerations, we show below that there is an open region of parameter values for which the second effect prevails over the first one and hence the optimal financial structure is a ring. Under such conditions, therefore, sparse connections are optimal.

Proposition 5 *Let $\widetilde{l}$ be distributed as a mixture of a Pareto distribution with parameter $2 > \gamma > 1$ and a Dirac distribution with all mass concentrated on $\bar{L} = 2(N-1) + 1$, with*

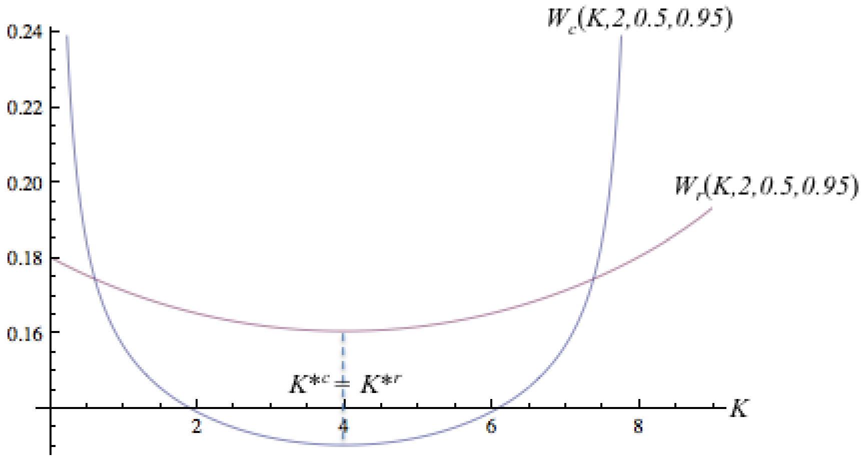

Figure 4: Expected mass of firms in a system with N firms indirectly hit by a b shock who default when this shock hits a component in the system, as given by the function $W_v(K,\cdot) \equiv \frac{K+1}{N} D_v(K,\cdot) + \frac{N-K-1}{N} D_v(N-K-2,\cdot)$, for the complete and ring structures $(v = c, r)$. The system consists of $N = 10$ firms divided into two components of size K and $N - K$. The parameter values are as in Figure 3, i.e. $\gamma = 2$, $\gamma' = 0.5$, $p = 0.95$.

weights respectively given by p and $1-p$. Then, for all values of N such that

$$N > 1 + \left(\frac{1}{4^{\gamma-1}} - \frac{1}{5^{\gamma}} + \frac{\frac{1}{2^{\gamma-1}} \frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}} \right)^{\frac{1}{2-\gamma}} \tag{25}$$

and provided that

$$\frac{(1-p)}{p} < (\gamma - 1) \left(\frac{1}{2(N-1)} \right)^{\gamma}, \tag{26}$$

there exists an open set of values of p such that the optimal financial structure is a single ring component.

The role of inequality (26) is to guarantee that the weight p on the Pareto distribution is sufficiently high that the optimal segmentation structure is determined by it and hence a *single* component is optimal for the completely connected structures. But then, given that there is also a non negligible probability that a large shock arrives that cannot be fully absorbed, some attempt at "controlling the induced damage" may be in order. And this is indeed what the ring achieves – a suitable compromise between the extent of *risk sharing* allowed by extensive connectivity (i.e. minimal segmentation) and the limits to wide *risk* *contagion* resulting from symmetrically strong connections within components.

4 Stability and optimality

We now examine the relationship between the optimal pattern of linkages derived in the previous section and the individual incentives to form those linkages. We explore, in other words, whether social welfare is aligned with the maximization of individual payoffs. To model the strategic considerations involved in the creation of networks structures, we will model a network-formation game. Since we have only explored the payoffs accruing to structures formed by complete components and rings, we need to restrict our analysis to such kinds of structures. For simplicity we concentrate on completely connected component structures in what follows. The network-formation game is then assumed to be conducted as follows:

- Firms independently submit their proposals concerning the components to which they want to belong. Formally, a strategy of each firm i is a subset $S_i \subset N$, with the property that $i \in S_i$.
- A particular proposal S_i is accepted only when all the firms $j \in S_i$ make the same proposal, i.e. $S_j = S_i$. Formally, given a profile of strategies $\mathbf{S} \equiv (S_i)_{i \in N}$ a component S is established if and only if for all the firms $i \in S$, $S_i = S$. If the proposal S_i of firm i is rejected, that is if for some $j \in S_i$, $S_j \neq S_i$, firm i remains in isolation.

Given the above network-formation rules, a specific network $\Gamma(\mathbf{S})$ is induced by each strategy profile $\mathbf{S}$. As maintained throughout, the payoff of each firm i is decreasing in $\varphi_i(\Gamma(\mathbf{S}))$, its own default probability resulting from the network induced by $\mathbf{S}$. Formally, we can w.l.o.g. identify its payoff with $-\varphi_i(\Gamma(\mathbf{S}))$, the opposite of that probability.

In such a network-formation game, an undesirable feature of the standard concept of Nash Equilibrium is that it leads to a vast multiplicity of equilibrium networks, a consequence of the fact that the formation of any link induces a coordination problem between the two agents involved. (As an extreme illustration, note that the empty network can always be supported by a Nash equilibrium where every agent proposes nobody to link with.) To address this issue, it is common in the literature to consider a strengthening of the Nash equilibrium notion that reduces miscoordination by allowing sets of agents to deviate jointly (see e.g. Goyal and Vega Redondo (2007) or Calvó-Armengol and Ikiliç (2009)). In the context of our model, we shall capture this idea by means of the concept we label *Coalition-Proof Equilibrium* (CPE), where any group (i.e. coalition) of agents can coordinate their deviations:

DEFINITION 1 *A strategy profile* $\mathbf{S} \equiv (S_i)_{i \in N}$ *of the network-formation game defines a* Coalition-Proof equilibrium *(CPE) if there is* no subset *of firms* W *and a strategy profile for these firms,* $(S'_j)_{j \in W}$*, such that*[21]

$$\forall i \in W, \quad \varphi_i \left[\Gamma \left((S'_j)_{j \in W}, (S_k)_{k \in N \setminus W} \right) \right] < \varphi_i[\Gamma(\mathbf{S})].$$

The *CPE* configuration is immune not only to individual but also to joint profitable deviations by any arbitrary set of firms, so it is obviously a refinement of the standard notion of Nash Equilibrium. A requirement typically imposed on coalition-based notions of equilibrium is that the coalitional deviations considered should be robust, in the sense of being themselves immune to a subcoalition profitably deviating from it. This requirement has no bite for our analysis since we can show that all the profitable deviations that need to be allowed are robust in the aforementioned sense.[22]

Consider first the case where the optimal completely connected structure consists of identical components of the size that minimizes the mass of defaults in a component – thus, in this case, the feasibility constraint $\sum_{i=1}^{C}(K_i + 1)/N = 1$ does not bind (as for instance in Proposition 2). Then it is immediate to verify that the optimal structure is also a *CPE*. In contrast, we show below (Proposition 6) that this is no longer necessarily true when the feasibility constraint binds.

To make the point, we analyze the *CPE* for the class of distributions considered in Section 3.2 and show that, in that context, there is typically a conflict between social opti-

[21] In line with what was postulated in Section 2, we shall continue to assume that, after any change in connections has been implemented, each firm involved in the change continues to distribute the fraction 0.5 of its own assets among all its neighbors (old and new) in a uniform manner.

[22] See Remark 2 in the Appendix.

mality and individual incentives. To understand the conflict, first recall from Proposition 4 that there is an open set of the parameter space for which, under the aforementioned class of shock distributions, the optimal completely connected structure involves a symmetric segmentation of the whole population into several components of equal size, $K^* + 1$. Moreover, $K^* + 1$ is typically smaller than the individually optimal component size $\hat{K} + 1$ that minimizes default *within* a given component.

This gap produces a conflict between individual and social incentives, which is reflected by the inefficiency of CPE configurations. In a CPE the outcome is asymmetric: individual incentives (supported by coalitional deviations) give rise to all but one component being of the individually optimal size $\hat{K} + 1$ and one other component of a size smaller than $K^* + 1$. But this, to repeat, cannot be socially optimal because standard convexity considerations favor a more balanced configuration where the sizes of all components are equal.

PROPOSITION 6 *Consider the same environment as in Proposition 4. Then, for $p \in (p_0, p_1)$, the socially optimal completely connected structure cannot be supported at a CPE of the network formation game. Among the completely connected structures, the only CPE configuration is asymmetric with all but one component displaying the size $\hat{K}$ that minimizes $D_c(K, \gamma, \gamma', p)$ and one component of a lower size.*

The basis for the above result is two-fold. First, it is clear that no component prevailing at a *CPE* can be larger than $\hat{K} + 1$, the size that is individually optimal. For, if it were, an obvious coalitional deviation would be available. Moreover, if there is any component of a size lower than $\hat{K} + 1$, there can be no more than one. For otherwise a profitable deviation towards a larger component would be admissible for a suitable coalition. In this context, the source of the conflict between social and individual optimality lies in the externality that is imposed by the fact that the feasibility constraint must hold in the aggregate across all components. Therefore, as firms in any component strive for the size that is individually (or component-wise) optimal, this will generally force other firms to remain in a component that is too small to share risk efficiently.

As a simple illustration of the problem, refer back to Example 2, where the efficient configuration involves two completely connected components of common size equal to $K^* +$

$1 = 5$, while the individually optimal size for each component is $\hat{K}^c + 1 = 6.65$, the value that minimizes $D_c(K, 2, 0.5, 0.95)$. In contrast the only *CPE* configuration is asymmetric with two components of sizes 6.65 and 3.35, the latter being a "residual" group. The example also makes clear that this is not the outcome of an *integer problem* in the sense that it does not disappear when the size of the economy grows large,[23] but rather to the presence of a feasibility constraint on what are the admissible components, within which risk is shared among firms, a constraint not internalized by individual firms when deciding which component to join.

The existence of a conflict between efficiency and strategic stability is of course hardly novel nor surprising in the field of social networks (see e.g. Jackson and Wolinsky (1996) for an early instance of it). For, typically, the creation or destruction of any link between two agents impose externalities on others that are not internalized by the two agents involved in the linking decision. In the context of risk-sharing, this tension has been studied in a recent paper by Bramoullé and Kranton (2007) – hereafter labeled BK – and it is interesting to understand its differences with our approach. We close, therefore, this section with a brief comparison of the two models.

BK consider an environment consisting of a finite number of agents affected by i.i.d. income shocks. Linkages generate risk sharing in a way similar to that of our model, except in two important respects: (a) risk-sharing is complete (i.e. uniform) across all members in a component; (b) there is no risk of contagion, so the size of optimal components is just limited by the fact that links are assumed to be costly. Focusing on the notion of strategic stability (which is weaker than ours),[24] BK are also interested in comparing efficient and equilibrium configurations. They find that whenever equilibrium structures exist (not always), there are at most two asymmetric components, with sizes smaller than the optimal one.

The contrast between BK's conclusions and ours derives from the nature of the externality in the two cases. In BK, given that the cost of any new link is borne only by the two agents involved, the equilibrium induces an underinvestment in link formation (which is a "public good"). Instead, in our case there are no linking costs, so the nature of the

[23]One can easily renormalize the parameters to construct examples where $\hat{K}^c + 1 = 665$ when $N = 1000$.

[24]Strategic stability allows only for coalitions of at most two players and rules out as well the simultaneous creation and destruction of links. See Jackson and Wolinsky (1996) for details.

externality that is not internalized is quite different. It has to do with the fact that, when firms deviate to reach a larger component of individually optimal size, they do not internalize that the firms that are left behind will be forced to components that are inefficiently small. Thus, in the end, it is the need to meet an overall feasibility constraint imposed by the finite measure of the population that typically generates inefficiencies.

5 Asymmetric environments

So far we have concentrated the discussion on a situation where all firms in the system are identical. Although this allows us to obtain analytical results and gather intuition, it is important to extend our analysis to situations where firms are significantly different in size or in their shock distribution, as often happens in the real world. To this end, we shall simplify the analysis and focus our attention in the next subsection on the case where structures are completely connected. In the following subsection we also allow for some asymmetry in network structures. Under these circumstances, there is no real gain in using the continuum approximation of the patterns of firms' exposure and thus we revert to the original specification, where firms are conceived as discrete entities (which can be of different types and sizes, as we explain below).

5.1 Shock and size asymmetry

To begin with, consider the case where all firms have equal resources R in the absence of shocks but they differ in the distribution of the shocks that may hit their project's returns. More precisely, suppose that the N firms are partitioned into subsets $N_1, ...N_n$, so that $N = \cup_{i=1}^{n} N_i$, and for every firm $j \in N_i$ the big shock follows a distribution F_{N_i} with common expected value for all $i \in \{1, ..., n\}$.

For each $i = 1, .., n$ we denote by $D_c(K, \Phi_{N_i})$ the expected number of firms who default in a (complete) component of size $K+1$ when indirectly hit by a b shock hitting a firm j belonging to subset N_i . Note that, since all firms are of equal size, each firm has the same capacity of absorbing shocks. This is what implies that the function $D_c(\cdot)$ depend only on the size of the component (as determined by K) and the distribution Φ_{N_i} of the shock hitting a firm j, and not also of the types of other firms in the component. In particular, each firm retains a fraction $1/2$ of its own project and acquires the same exposure $1/(2K)$ to shocks hitting any other firm in the component[25]. We thus have:

$$D_c(K, \Phi_{N_i}) = K\left[1 - \Phi_{N_i}(2K)\right]. \tag{27}$$

Let $K^*_{N_i}$ be the minimizer in K of this function.

We are interested in identifying the optimal segmentation structure, which now requires determining not only the number of firms that should lie in each component but also the composition of each component in terms of the different types of firms included in it. One important concern will be to understand whether firm matching within components should be assortative or dissortative – that is, whether components should be more or less homogeneous. In this context, therefore, a structure must specify a set of components $\mathcal{C} \equiv \{C_j\}_{j=1,\ldots,m}$ and, for each component j, the corresponding type distribution $\{N^j_i\}_{j=1,\ldots,n}$, where $C_j = \cup^n_{i=1} N^j_i$ and each N^j_i stands for the subset of firms in N_i that belong to component j.

Proposition 7 *Suppose that, for every i, the cardinality $|N_i|$ of the set of firms of type i is a multiple of $K^*_{N_i}+1$. Then, the optimal (completely connected) structure has all firms belonging to homogeneous components, in each of which firms are all of the same type, and $K^*_{N_i}+1$ is the common size of every component consisting of firms of any given type i.*

The intuition for this result is as follows. For each i, conditional on the b shock hitting a firm of type i, the optimal arrangement is to have all components where firms i are present with $K^*_{N_i}+1$ firms in them, as $K^*_{N_i}$ is the value that minimizes the function $D_c(K, \Phi_{N_i})$ given in (27). If we had some heterogeneous component, with firms of type j, in addition to i in it, its size could not be set to equal *simultaneously* the (generally different) optimal sizes for the cases of shocks hitting type j and type i firms. Hence the expected number of defaults could be reduced by rearranging firms within homogenous components (if this is feasible), as this would allow to attain the minimum of $D_c(K, \Phi_{N_i})$ for each i. Of course,

[25] This is a consequence of the fact that, even though the risks are different, the exchange of assets is still one for one for every pair of firms.

a key feature required for this conclusion to hold is that, as already noticed, the expected number of indirect defaults in a component only depends on the type i of the firm directly hit by the shock and the size of the component. That is, the indirect effect of the shock does *not* depend on the distribution of types among the firms in the component not directly hit by the shock. Because of this intuition it seems clear that although we have done the analysis explicitly for completely connected components, the same result would hold for other symmetric structures, where all components were, say rings.

Next, we turn our attention to the case where firms are heterogeneous in size. In light of the previous result, which establishes the optimality of extreme assortativity by distributional types, we now concentrate on environments where all firms are of the same type as far as the distribution of the shock hitting them is concerned. For simplicity, let us consider a situation with just two possible sizes. On the one hand, there are firms of unit size, identical to the ones we have been considering so far. On the other hand, there are firms of size $\beta > 1$, such larger size having the following two implications. First, the return on the projects of these larger firms when no shock affects them is βR, so it is scaled up by β compared to those of the smaller firms. (Naturally, we also assume that the same factor β applies to the value of their liabilities.) Second, the larger firms are supposed to face a probability of being directly hit by a shock that is β times larger (that is, equal to βq) Hence, in this case, we make an assumption that is polar to the case considered before (and thus leads to a complementary analysis): size affects the probability of arrival of the shocks, *not* their distribution. In a sense, therefore, one can view a large firm as a full merge of β small firms of the sort we have been considering so far.

Of course, to ensure the equal exposure to the shocks hitting any of the other firms to which a given firm is linked, in presence of heterogeneity in firms' sizes the fractions held need to be suitably rescaled: a fraction held by a firm of size 1 of the outstanding amount of claims of a firm of the same size is in fact equivalent to $1/\beta$ times the same fraction held in the claims of a firm of size β, and viceversa. Let us denote by N_1 the subset of firms of size 1, and by N_β the subset of firms of size β, while N_1^j and N_β^j stand for the corresponding subsets of small and big firms within a component C_i. The size of this component is then given by $K_j + 1 = \left|N_1^j\right| + \beta \left|N_\beta^j\right|$ where $|\cdot|$ stands for the cardinality of the set in question. The expected number of defaults in the component if the firm hit is small (i.e. of size 1) is:[26]

$$\begin{aligned} D_c(K_j, 1) &= \left(\left|N_1^j\right| - 1\right) \Pr\left\{\frac{\widetilde{l}}{2K_j} > 1\right\} + \beta \left|N_\beta^j\right| \Pr\left\{\frac{\beta\widetilde{l}}{2K_j} > \beta\right\} \\ &= K_j \Pr\left\{\frac{\widetilde{l}}{2K_j} > 1\right\}, \end{aligned} \tag{28}$$

while if the firm hit is large (of size β) the expected number of defaults is

$$\begin{aligned} D_c(K_j, \beta) &= \left|N_1^j\right| \Pr\left\{\tfrac{\widetilde{l}}{2(K_j+1-\beta)} > 1\right\} + \beta\left(\left|N_\beta^j\right| - 1\right) \Pr\left\{\tfrac{\beta\widetilde{l}}{2(K_j+1-\beta)} > \beta\right\} \\ &= (K_j + 1 - \beta) \Pr\left\{\tfrac{\widetilde{l}}{2(K_j+1-\beta)} > 1\right\}. \end{aligned} \tag{29}$$

For each $s \in \{1, \beta\}$, define K_s^* as the minimizer in K of $D_c(K, s)$. The total expected number of defaults in a system divided into m components when a b shock hits a firm is then:

$$\begin{aligned} & \sum_{j=1}^{m} \frac{K_j+1}{N} \left(\frac{\left|N_1^j\right|}{K_j+1} D_c(K_j, 1) + \frac{\left|N_\beta^j\right|}{K_j+1} D_c(K_j, \beta) \right). \\ = & \sum_{j=1}^{m} \left(\frac{\left|N_1^j\right|}{N} D_c(K_j, 1) + \frac{\left|N_\beta^j\right|}{N} D_c(K_j, \beta) \right) \\ \geq & \sum_{j=1}^{m} \left(\frac{\left|N_1^j\right|}{N} D_c(K_1^*, 1) + \frac{\left|N_\beta^j\right|}{N} D_c\left(K_\beta^*, \beta\right) \right) \\ = & \ D_c(K_1^*, 1) \frac{|N_1|}{N} + D_c\left(K_\beta^*, \beta\right) \frac{|N_\beta|}{N} \end{aligned}$$

This establishes the following result:

PROPOSITION 8 *Suppose that* $|N_1|$ *and* $|N_\beta|$ *are a multiple, respectively, of* K_1^*+1 *and* K_β^*+1. *Then the optimal (completely connected) structure is such that all firms are in components*

[26] To derive the expression below note that, as a result of the fact that a small firm exchanges claims at a rate $1/\beta$ with the big firm, each small firm ends up having a share $1/2\,(K_i + 1 - \beta)$ of the outstanding amount of claims of the big firm, while the big firm has a share $\beta/2K_i$ of the claims of each small firm.

with all firms of the same size (either small or large) and the common size of every component consisting of small (resp. large) firms is $K_1^* + 1$ $(K_\beta^* + 1)$.

It is worth noting that the same result would hold if firm size, 1 or β, instead of scaling up the probability were to scale the size of the b shock hitting firms – that is, if the probability that a firm of size 1 is hit by a shock of magnitude l were the same as the probability that a firm of size β is hit by a shock of size βl. It can be easily checked that, even though the counterpart of expressions (28) and (29) would be different, they would still retain the property of being independent of the composition between small and large firms in the component. And, as explained before, this is the key feature that yields the above conclusion, that the optimal configuration involves the segmentation of the population into homogeneous components, with all firms of equal size. The intuition is similar to the one discussed when motivating Proposition 7: allowing for heterogeneous components would mean failing to take advantage of the generally one-to-one correspondence between component size and type – be it associated to size or shock distribution – of the firm hit by the shock, that assures optimality.

5.2 Asymmetry in structure

Once we allow for different sizes of firms, we may also examine new types of structures in which large and small firms may play asymmetric roles. The key question we now study is whether it is optimal that firms of different size have a different position in the network, and hence a different role in the pattern of risk sharing trades. To this end, our analysis will focus on comparing two structures: (a) one *symmetric*, where components are completely connected and each component involves only firms of identical size; (b) another one asymmetric, where components have a *"star" structure*, consisting of a large firm that acts as a central hub and is directly connected to various small firms.

More precisely, we consider for simplicity the case where there are only two large firms and 2β small ones. Hence in structure (a) we have two completely connected components of the same total size: one with the two large firms; the other with the 2β small firms. In contrast, in structure (b) we have two identical star components, each consisting of β small firms that are solely connected to a large firm (that is, the large firm acts as a hub and there are β spokes of unit size). Hence in this case each component is heterogeneous and not fully connected.

The first step is to specify the pattern of risk exposure in each of the two cases. In structure (a), the situation is analogous to that of the completely connected structures considered in Subsection 2.2. The exposure pattern can then be described by a matrix A_K of the form specified in (7) for a component of size $K+1$, with $K=1$ and $K=2\beta-1$ for the complete components consisting of large and small firms, respectively.

In a star structure (b) we face the new problem that not only firms are heterogeneous within each component, but also the structure of the component is asymmetric. To ensure some consistency in the pattern of linkages among firms in this situation, it is then convenient to consider the one emerging from a repeated exchange of assets among firms, according to the process described in Example 1. The pattern of asset exchanges among the firms in each star component is:[27]

$$B^s = \begin{pmatrix} \theta & (1-\theta)/\beta & (1-\theta)/\beta & \cdots & (1-\theta)/\beta \\ (1-\theta) & \theta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\theta) & 0 & 0 & \cdots & \theta \end{pmatrix}$$

The entries of the matrix B^s reflect the fact that, for the reasons explained in the previous section, the exchange of assets among firms of different size is no longer one for one: the large firm (indexed by $j=1$) must offer only a share $(1-\theta)/\beta$ of its assets for a larger share $(1-\theta)$ in the assets of small firms (indexed by $j=2,3,...,\beta+1$). Iterating once these trades we obtain the following pattern of exposure, $A^s = (B^s)^2$, where each firm has

[27] We dispense here for notational simplicity with any reference to the component size ($K=\beta+1$).

a non-zero exposure to any other firm in the component

$$A^s = \begin{pmatrix} \theta^2 + (1-\theta)^2 & 2\theta(1-\theta)/\beta & (1-\theta)/\beta & \cdots & (1-\theta)/\beta \\ 2\theta(1-\theta) & \theta^2 + (1-\theta)^2/\beta & (1-\theta)^2/\beta & \cdots & (1-\theta)^2/\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\theta(1-\theta) & (1-\theta)^2/\beta & (1-\theta)^2/\beta & \cdots & \theta^2 + (1-\theta)^2/\beta \end{pmatrix}.$$

From the above expression for A^s, denoting by a^s_{ij} the ij entry of A^s, we see that, for each value of θ, the fraction of claims held by each small firm on its own project:

$$a^s_{ii} = \theta^2 + (1-\theta)^2/\beta, \text{ for } i > 1, \tag{30}$$

is smaller than the analogous fraction held by the large firm:

$$a^s_{11} = \theta^2 + (1-\theta)^2. \tag{31}$$

Naturally, the difference $a^s_{11} - a^s_{ii} > 0$ grows larger with β. This is simply a reflection of the fact that the higher is the size asymmetry between large and small firms the lower the share of its own project that the large firm needs to trade for any given share on the project of a small firm. The value of θ will be set so that the degree of externalization of risk of *every* firm in the system (large or small) is not larger than 1/2, as in Subsection 2.2, that is each firm retains a fraction of least 1/2 of the claims to the yields of its own project. From (30) and (31), it is clear that this happens when $a^s_{ii} = 1/2$ and $a^s_{11} = 1/2 + (1-\theta)\frac{\beta-1}{\beta}$.

As in Section 2.1, any firm i defaults when a shock l hits the project of firm j in the same component (possibly $i = j$) and the firm's exposure to the shock, $a^s_{ij}l$, exceeds the firm's net returns. This now amounts to a different condition if the firm under consideration is big or small, since each firm has a different "return buffer": if i is small, default occurs when $a^s_{ij}l > 1$; instead, if i is large the condition is $a^s_{ij}l > \beta$.

The next result compares the performance of structures (a) and (b), for different values l of the magnitude of the b shock hitting a randomly selected firm:

PROPOSITION 9 *Assume $\beta > 2$ and let a^s_{11} be the solution of (30), (31) when $a^s_{ii} = 1/2$.*

Then, if

$$\frac{1}{1-a_{11}^s} < l \leq 2\beta, \tag{32}$$

expected defaults are lower in structure (a) with homogeneous components than in structure (b) with star components, while if

$$2(2\beta - 1) < l \leq \max\left\{\frac{\beta-1}{a_{11}^s - 1/2}, \frac{\beta^2}{1-a_{11}^s}\right\} \tag{33}$$

the opposite conclusion holds.

First, the superiority of the symmetric (completely connected) structure in the range given by (32) is easy to explain. In the star structure a large firm (the hub) engages in risk sharing trades with a set of smaller firms (the spokes). This ends up limiting the possibilities of risk sharing when a shock hits a large firm because, in order to match the lower value of the assets of the smaller firms, the large firm is forced to hold more of its own assets than in the symmetric structure. As a consequence, when a shock of an intermediate size as in (32) hits a large firm, it triggers the default of the small firms in the star structure as well as, possibly, that of the large firm. Instead such a shock hitting the large firm would be absorbed with no defaults in a symmetric structure, due to the enhanced risk sharing possibilities it would enjoy when connected to a firm of the same size.

On the other hand, it is precisely this limitation to the risk sharing possibilities in a star structure that protects the hub as well as the other spokes against the shocks hitting a small firm. As a consequence, the star structure performs better than the symmetric structure for shocks of larger size, as in (33). In a symmetric component, these shocks are so large that they trigger the default of all the firms linked to a firm directly hit by the shock. In contrast, in a star component none of these firms defaults when the shock hits one of the spoke firms – only the small firm hit defaults. In this case, therefore, the large firm acts a sort of buffer, preventing a sizable fraction of the shock to spread and cause further defaults in the component.

6 Conclusion

We have proposed a stylized model to study the problem that arises when firms need to share resources to weather shocks that can threaten their survival, but by so doing they become exposed to the risk coming from those same connections that help them in the time of need. Depending on the characteristics of the shock distribution, a wide variety of different configurations turn out to be optimal. For example, maximal segmentation in small groups is optimal if big shocks are likely, while very large groups are optimal when most shocks are of moderate magnitude. There are also conditions, however, when an intermediate group size is optimal or when groups should be large but display some internal "detachment" (i.e. sparse connectivity).

The former consideration pertains to social optimality, i.e. to the minimization of the default rate within the whole system. We have also explored whether such social optimum is aligned with individual optimality. And we have seen that, in general, there is a conflict between strategic incentives and social welfare. This tension arises from the fact that firms have always an incentive to form connected components of the size that minimizes the default probability of their members, thus ignoring the negative externality this behavior imposes on other firms. Finally, we have found that when heterogeneity among firms is allowed but network components are taken to be fully and symmetrically connected, optimality is achieved under perfect assortativity. But if the network structure can display some asymmetries (e.g. a "large central" agent can act as a hub), it may be optimal to take advantage of this possibility as a *firebreak* under certain conditions.

The present analysis represents a first step in studying the welfare implications of alternative risk-sharing network structures, as well as their (in)consistency with individual incentives. By restricting attention to suitable families of network structures, we have arrived at clear-cut conclusions as to what kind of segmentation, sparseness, or asymmetries in connections attain optimality. The basic insights obtained from this analysis could inform the regulation of financial systems, which are indeed subject to some of the key mechanisms contemplated by our theoretical framework. Should one separate commercial and investment banking? Should the financial systems of different regions be insulated from each other, or should overall integration be pursued? Is it advisable to allow for some large central institutions to have a buffer role in mitigating the shocks affecting smaller ones? And, in the end, how effectively can we trust any prescription along these lines to be indeed implemented by the economic agents themselves (i.e. to be compatible with their own individual incentives)? These are some of the questions that will naturally and crucially arise in the design and regulation of financial systems in the real world.

Even though our stylized model cannot possibly provide *direct* policy advice on the previous questions, it does highlight some of the key considerations that should underlie a systematic analysis of them. As mentioned in the Introduction, the companion paper by Loepfe et al. (2013) extends our analysis through numerical methods to environments with more realistic features and verifies the robustness of our main results. More precisely, it finds that the optimality of forming smaller/less dense networks when shocks come from distributions that put more weight on large shocks extends even when one allows for the possibility of multiple simultaneous shocks, for other kinds of distributions and for networks that present heterogeneity in the firm sizes and the distribution of links among the different agents. In a different vein, they also find similar effects on the real-life network of corporate control studied by Battiston *et al.* (2012), where the consequences of link removal on the vulnerability of the system (increasing or decreasing it) happens to depend on the nature of the shock distribution as intuitively suggested by the theory.

But, of course, a proper discussion of the risk-sharing and contagion phenomena in the real world must also account for many important dimensions that our model abstracts from. One of them is the consideration of problems of moral hazard and, in general, asymmetries of information that have been repeatedly singled out as a key factor underlying the recent financial crisis. Any improvement in the regulatory framework should tackle these serious problems carefully. Finally, another point that is worthy of further research is the integration of the risk-management decisions studied here with the considerations of inter-agent cooperation and exploitation of synergies that also underlie economic connections in the real world. A proper study of all these crucial issues will demand a richer theoretical framework and a more powerful methodology, the development of which can hopefully build in a fruitful manner upon the present effort.

Appendix

Proof of Lemma 1: Let $M_F(qN\pi_b)$ denote the function specifying the value of M that satisfies the required arbitrage condition (4) under financial structure F, given by a pattern of interaction $\nu \in \{c, r\}$ and component sizes $K_1, ..., K_n$ all displaying the pattern ν, when the probability of the arrival of a b shock is $qN\pi_b$. Substituting equation (21) into (4), $M_F(qN\pi_b)$ is defined implicitly by the following equation:

$$M_F(qN\pi_b)\left[1 - qN\pi_b\left(\frac{1}{N}\Pi + \sum_{i=1}^{C}\frac{K_i+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{l}; K_i, M_F(qN\pi_b))}{N}\right)\right] = r.$$

Since the function $\mathbb{E}g_\nu(\widetilde{l}; K_i, M_F(qN\pi_b))$ is bounded for each $\nu \in \{c, r\}$, as we see from (20), (18), $M_F(qN\pi_b)$ is continuous in $qN\pi_b$ at $qN\pi_b = 0$. Hence, if structure F_1 yields a higher expected mass of defaults when indirectly hit by a b shock than structure F_2, i.e.

$$\sum_{i=1}^{C^{F_1}}\frac{K_i^{F_1}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{l}; K_i, M_{F_1}(0))}{N} > \sum_{i=1}^{C^{F_2}}\frac{K_i^{F_2}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{l}; K_i, M_{F_2}(0))}{N},$$

then we also have, for $qN\pi_b$ sufficiently small,

$$\sum_{i=1}^{C^{F_1}}\frac{K_i^{F_1}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{l}; K_i, M_{F_1}(qN\pi_b))}{N} > \sum_{i=1}^{C^{F_2}}\frac{K_i^{F_2}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{l}; K_i, M_{F_2}(qN\pi_b))}{N},$$

■

Proof of Proposition 6: Let $\mathbf{S} \equiv (S_i)_{i\in N}$ be a *CPE* of the network-formation game with completely connected components. Denote by C^S the number and by $\{K_j+1\}_{j=1}^{C^S}$ the sizes of the different components of the *CPE* network $\Gamma(\mathbf{S})$. Recall that $\hat{K}$ denotes the unique value that minimizes $D_c(K, \gamma, \gamma', p)$ (c.f. the proof of Proposition 4).

The proof has two main steps. We show first that for each component $j = 1, .., C^S$ we have $K_j \leq \hat{K}$. That is, all components in the *CPE* have size smaller or equal than the individually optimal one. Suppose not: $K_j > \hat{K}$ for some j. Choose then a subset $\mathcal{D}$ of the firms belonging to component j whose measure satisfies $K_j - \hat{K} \geq |\mathcal{D}| > 0$. Since the function $D_c(K, \gamma, \gamma', p)$ is increasing in K for $K > \hat{K}$, $D_c(K_q - |\mathcal{D}|, \gamma, \gamma', p) <$

$D_c(K_q, \gamma, \gamma', p)$. This implies that all the firms who are in component j except those in $\mathcal{D}$ could enjoy a lower default probability by proposing a component with all the firms in j except those with the firms in $\mathcal{D}$. This establishes the desired conclusion.

Next, we show that at a *CPE* we have $K_{\widetilde{h}} < \hat{K}$ for some $\widetilde{h} \in \{1, 2, ..., C^S\}$ and $K_j = \hat{K}$ for all $j \neq \widetilde{h}$. We proceed again by contradiction. Suppose that there are two components of $\Gamma(\mathbf{S})$, j and q, such that $0 < K_j < K_q < \hat{K}$. Pick then a subset of firms belonging to the first component: $\mathcal{D}_j \subset K_j$, of measure $|\mathcal{D}_j| \leq \hat{K} - K_q$. Consider the following joint deviation for the firms in $\mathcal{D}_j$ as well as for all those in component q from their strategies in $\mathbf{S}$. Each of the firms in $\mathcal{D}_j$ together with the firms in component q propose a component consisting of all of them (and only them). Then, we have:

$$D_c(K_q + |\mathcal{D}_j|, \gamma, \gamma', p) < D_c(K_q, \gamma, \gamma', p) < D_c(K_j, \gamma, \gamma', p)$$

Hence all firms involved benefit from the deviation, which contradicts the fact that at a *CPE* we have $0 < K_j < K_q < \hat{K}$. This contradiction establishes the above claim and completes so the proof of the Proposition. ■

Remark 2 *Note that the deviations contemplated in the proof of Proposition 6 are "internally consistent" in the following sense. Given any set of firms that find optimal to deviate, this deviation is itself in the interest of any subset of this set that might reconsider the situation. This requirement (commonly demanded in the game-theoretic literature for coalition-based notions of equilibrium) is clearly satisfied in our case since, by refusing to follow suit with the deviation, any firm in this subset can only lose, because of the monotonicity of the function* $D_c(.)$.

Proof of Proposition 7: If a component i is hit a by a b shock, the expected number of defaults among firms not directly hit by it is:

$$\sum_{j=1}^{n} \frac{\left|N_j^i\right|}{K_i + 1} D_c\left(K_i, \Phi_{N_j}\right),$$

where $K_i + 1$ is as usual the size of component i, in this case given by $\sum_{j=1}^{n} \left|N_j^i\right|$. Then the total expected number of defaults in the system when a b shock hits a firm is

$$\begin{aligned} & \sum_{i=1}^{I} \frac{K_i+1}{N} \sum_{j=1}^{n} \frac{\left|N_j^i\right|}{K_i+1} D_c\left(K_i, \Phi_{N_j}\right). \\ = & \sum_{i=1}^{I} \sum_{j=1}^{n} \frac{\left|N_j^i\right|}{N} D_c\left(K_i, \Phi_{N_j}\right). \end{aligned}$$

Exchanging the summation indices, we get

$$\begin{aligned} \sum_{j=1}^{n} \sum_{i=1}^{I} \frac{\left|N_j^i\right|}{N} D_c\left(K_i, \Phi_{N_j}\right) & \geq \sum_{j=1}^{n} \sum_{i=1}^{I} \frac{\left|N_j^i\right|}{N} D_c\left(K_{N_j}^*, \Phi_{N_j}\right) \\ & = \sum_{j=1}^{n} D_c\left(K_{N_j}^*, \Phi_{N_j}\right) \sum_{G_i=1}^{I} \frac{\left|N_j^i\right|}{N} = \sum_{j=1}^{n} \frac{\left|N_j\right|}{N} D_c\left(K_{N_j}^*, \Phi_{N_j}\right). \end{aligned}$$

Since $|N_j|$ is a multiple of $K_{N_j}^* + 1$ the lower bound

$$\sum_{j=1}^{n} \frac{\left|N_j\right|}{N} D_c\left(K_{N_j}^*, \Phi_{N_j}\right)$$

is also the expected number of defaults if every firm is part of a group of optimal size $K_{N_j}^* + 1$ with all firms of the same type. ■

Proof of Proposition 9: To prove the result, we compute the expected number of bankruptcies associated to the two structures (star and symmetric) for all possible levels of the b shock. Denote by $G_{star}(l)$ and $G_{sym}(l)$ the functions that specify the total size of all[28] the firms who default, resp. for the star and the symmetric structure, as a function of the magnitude l of the b shock.

Let us begin by determining the total size of firms defaulting if the shock hits a *small firm* and the structure of the component to which it belongs is a star. For any given value

[28]Note that, in contrast to the functions $g_\nu(L; K, M)$, $\nu = c, r$, introduced in Section 2.2, the functions $G_{star}(\cdot)$, $G_{sym}(\cdot)$ describe the number of *all firms* who default, that is including the firm that is directly hit by the shock L. Since the degree of risk externalization, as we noticed, may now differ across different structures, so does the probability that a firm directly hit by a shock defaults.

of l, it is given by the following function[29]:

$$G_{star}(l;s)=\begin{cases}0 & \text{for } l\leq \frac{1}{a^s_{ii}}=2\\ 1 & \text{for } 2<l\leq \min\left\{\frac{\beta-1}{a^s_{11}-1/2},\frac{\beta^2}{1-a^s_{11}}\right\}\\ \begin{matrix}\beta \text{ if } \frac{\beta-1}{a^s_{11}-1/2}\leq\frac{\beta^2}{1-a^s_{11}}\\ 1+\beta \text{ otherwise}\end{matrix} & \text{for } \min\left\{\frac{\beta-1}{a^s_{11}-1/2},\frac{\beta^2}{1-a^s_{11}}\right\}<l\leq\max\left\{\frac{\beta-1}{a^s_{11}-1/2},\frac{\beta^2}{1-a^s_{11}}\right\}\\ 2\beta & \text{for } l>\max\left\{\frac{\beta-1}{a^s_{11}-1/2},\frac{\beta^2}{1-a^s_{11}}\right\}\end{cases}$$

Instead, if the structure is still a star but the shock hits a *large firm* (i.e. the hub), the expected number of defaults is given by the following function[30]:

$$G_{star}(l;\ell)=\begin{cases}0 & \text{for } l\leq\frac{1}{1-a^s_{11}}\\ \beta & \text{for } \frac{1}{1-a^s_{11}}<l\leq\frac{\beta}{a^s_{11}}\\ 2\beta & \text{for } l>\frac{1}{1-a^s_{11}}\end{cases},$$

Ex ante, a shock hitting a component of the network has the same probability of striking a large firm or a small firm. Hence, $G_{star}(l)=\left(G_{star}(l;s)+G_{star}(l;\ell)\right)/2$.

Consider next the case where the structure is symmetric, with two completely connected components, the first one with the two large firms, the second one with the 2β small firms. In this case, since there is no asymmetry within each component, every firm retains a fraction exactly equal to $a_{ii}=1/2$ of claims to the returns of its own project. The off diagonal terms of the exposure matrix are then equal to $1/2$ for the component with the two large firms and $1/[2(2\beta-1)]$ for the second one, with 2β firms. Since the shock reaches with equal probability each of the two components, the expected number of defaults is given by the following function:

[29] Recall that a small firm i defaults when a shock hits a firm j if and only if $\tilde{a}_{ij}L>1$, while a large firm k defaults if and only if $\tilde{a}_{kj}L>\beta$.

[30] The upper and lower bounds on the value of a^s_{11} established in Lemma 3 in Appendix B imply that $\beta/a^s_{11}>1/(1-a^s_{11})$.

$$G_{sym}(l) = \begin{cases} 0 & \text{for } l \leq 2 \\ \frac{1}{2} & \text{for } 2 < l \leq 2\beta \\ \frac{1}{2} + \frac{1}{2}2\beta & \text{for } 2\beta < l \leq 2(2\beta - 1) \\ 2\beta & \text{for } l > 2(2\beta - 1) \end{cases}$$

where it can be easily verified that $2 < 2\beta < 2(2\beta - 1)$.

A straightforward comparison of the functions $G_{star}(l)$ and $G_{sym}(l)$, noting that $\frac{\beta}{a_{11}^s} < 2\beta$ and $\frac{\beta}{a_{11}^s} < 2(2\beta - 1) < \min\left\{\frac{\beta-1}{a_{11}^s - 1/2}, \frac{\beta^2}{1-a_{11}^s}\right\}$, yields then the claim in the proposition. ■

The remaining proofs of Lemmas 2, 3 and Propositions 3, 4 and 5, as well as some further details of the proof of Proposition 9 can be found in Appendix B[31].

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7 Appendix B (for online publication)

Proof of Lemma 2: Rearranging terms in (23) and simplifying we get:

$$D_r(K,\gamma) = \left(K\left(\frac{1}{\gamma+1}\right) - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma} + \frac{1}{K-1}\frac{1}{(\gamma+1)}\left(\frac{1}{2}\right)^{\gamma-1}, \tag{34}$$

and hence

$$\begin{aligned} &\tfrac{\partial D_r}{\partial K}(K,\gamma) = -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\ &+\left(K\left(\tfrac{-\gamma}{\gamma+1}\right)\tfrac{1}{K+1} + \tfrac{2}{K-1}\tfrac{1}{K+1}\tfrac{\gamma}{\gamma+1} + \left(\tfrac{1}{\gamma+1}\right) + \tfrac{2}{(K-1)^2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{K+1}\right)^{\gamma}. \end{aligned} \tag{35}$$

Now note that the inequality $\partial D_r(K,\gamma)/\partial K > 0$ is equivalent to:

$$\frac{2\,(K-1)}{K+1}\gamma + (K-1)^2 + 2 > (K+1)\left(\frac{K+1}{2}\right)^{\gamma-1} + \gamma K\frac{(K-1)^2}{K+1},$$

or

$$(K-1)^2\left(1-\frac{\gamma K}{K+1}\right) + 2\left(1+\gamma\frac{K-1}{K+1}\right) > (K+1)^{\gamma}\frac{1}{2^{\gamma-1}},$$

which can be rewritten as

$$(K-1)^2 + 2 + \gamma\,(K-1)\,(2-K) > (K+1)^{\gamma}\frac{1}{2^{\gamma-1}}. \tag{36}$$

So, using the identities

$$(K-1)^2 + 2 + (K-1)\,(2-K) = K+1$$

and

$$(K+1)^{\gamma}\frac{1}{2^{\gamma-1}} = 2\left(\frac{K+1}{2}\right)^{\gamma}$$

we can equivalently write (36) as follows:

$$\frac{K+1}{2} - \left(\frac{K+1}{2}\right)^{\gamma} - \frac{1}{2}(1-\gamma)\,(K-1)\,(2-K) > 0.$$

Denote by $\Xi(K,\gamma)$ the term on the left hand side of the previous inequality, conceived as a function of K and γ. Then, to complete the proof, we establish the following property:

$$\forall K > 1, \quad \Xi(K,\gamma) \gtrless 0 \Leftrightarrow \gamma \lessgtr 1. \tag{37}$$

To show this property, note first that $\Xi(K,1) = 0$ for all K, so that $\frac{\partial D_r}{\partial K}(K,\gamma) = 0$ for $\gamma = 1$ and all K. On the other hand,

$$\begin{aligned} \frac{\partial \Xi}{\partial \gamma}(K,\gamma) &= -\left(\frac{K+1}{2}\right)^{\gamma} \ln \frac{K+1}{2} + \frac{1}{2}(K-1)(2-K) \\ &\leq -\ln \frac{K+1}{2} + \frac{1}{2}(K-1)(2-K), \end{aligned}$$

the inequality being strict for all $K > 1$. It is then easy to verify that the terms on the two sides of the above inequality are equal to 0 when $K = 1$ and the term on the right hand side is negative[32] for all $K > 1$, establishing (37) and hence also $\partial D_r(K,\gamma)/\partial K \gtrless 0 \Leftrightarrow \gamma \lessgtr 1$.

We conclude, as stated in the proposition, that the minimum of $D_r(K,\gamma)$ is attained at the maximum admissible value of K (i.e. $N-1$) when $\gamma > 1$, while it is attained at the lowest value of K (i.e. $K = 1$)[33] when $\gamma < 1$. This completes the proof of the lemma. ■

Proof of Lemma 3 Note first that the expression of $\partial D_r(K,\gamma)/\partial K$ obtained in (34) can be conveniently rewritten as follows:

[32]We have in fact

$$\frac{d\left(-\ln \frac{K+1}{2} + \frac{1}{2}(K-1)(2-K)\right)}{dK} = \frac{K+1-2K^2}{K+1} < \frac{2K(1-K)}{K+1} < 0$$

[33]To complete the argument we verify the claimed continuity property of $D_r(K,\gamma)$, as in (34), at $K = 1$:

$$\begin{aligned} &\lim_{K\to 1} D_r(K,\gamma) \\ =\ &\left(\frac{1}{2}\right)^{\gamma} \frac{1}{\gamma+1} - \lim_{K\to 1} \frac{1}{K-1} \frac{2}{\gamma+1} \left[\left(\frac{1}{K+1}\right)^{\gamma} - \left(\frac{1}{2}\right)^{\gamma}\right] \\ =\ &\left(\frac{1}{2}\right)^{\gamma} \left(\frac{1}{\gamma+1}\right) - \frac{-\gamma 2^{\gamma}}{(\gamma+1)4^{\gamma}} = \left(\frac{1}{2}\right)^{\gamma} = D_r(1,\gamma) \end{aligned}$$

$$\frac{\partial D_r}{\partial K}(K,\gamma) = -\frac{1}{(K-1)^2}\frac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1}$$
$$+\left(\frac{-K^2+K+2}{K-1}\frac{1}{K+1}\frac{\gamma}{\gamma+1}+\frac{1}{\gamma+1}+\frac{2}{(K-1)^2}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}$$
$$= -\frac{1}{(K-1)^2}\frac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1}$$
$$+\frac{1}{\gamma+1}\frac{K^2-2K+3}{(K-1)^2}\left(\frac{1}{K+1}\right)^{\gamma}-\frac{\gamma}{K+1}\left(K^2-K-2\right)\frac{1}{K-1}\left(\frac{1}{K+1}\right)^{\gamma}$$

Differentiating then again with respect to K yields:

$$\begin{aligned}\frac{\partial^2 D_r(1/2,K,\gamma)}{\partial K^2} &= 2\frac{\left(\frac{1}{2}\right)^{\gamma-1}}{(\gamma+1)(K-1)^3}-\frac{1}{\gamma+1}\frac{\left(\frac{1}{K+1}\right)^{\gamma}}{(K-1)^3(K+1)}\times \\ &\quad \left(K^3\left(-\gamma^2+\gamma\right)+2K^2\left(2\gamma^2-\gamma\right)+5K\left(-\gamma^2+\gamma\right)+4K+2(\gamma-1)^2+2\right) \\ &= \frac{1}{2^{\gamma}(\gamma+1)(K-1)^3}\left(4-\frac{2^{\gamma}}{(K+1)^{\gamma+1}}\left((K-1)^2K\left(\gamma-\gamma^2\right)\right.\right. \\ &\quad \left.\left.+2\gamma^2(K-1)^2+4\gamma(K-1)+4(K+1)\right)\right)\end{aligned}$$

Hence

$$\frac{\partial^2 D_r(1/2,K,\gamma)}{\partial K^2}<0$$

if and only if

$$G(K)\equiv\frac{\left(\frac{K+1}{2}\right)^{\gamma+1}}{\frac{(K-1)^2}{8}\left(K\left(\gamma-\gamma^2\right)+2\gamma^2\right)+\gamma\frac{(K-1)}{2}+\frac{(K+1)}{2}}<1 \qquad (38)$$

First, we observe that $G(1)=1$. Thus, to establish (38), it is enough to show that G is decreasing for all $K>1$. Letting $x\equiv K-1$ for notational simplicity, $\frac{dG(K)}{dK}<0$ if, and only if,

$$\frac{\frac{d}{dx}\left(\left(\frac{x}{2}+1\right)^{\gamma+1}\right)}{\left(\frac{x}{2}+1\right)^{\gamma+1}}<\frac{\frac{d}{dx}\left(\gamma x\left(\frac{x}{8}\left(x(1-\gamma)+1+\gamma\right)+\frac{1}{2}\right)+\frac{x}{2}+1\right)}{\left(\gamma x\left(\frac{x}{8}\left(x(1-\gamma)+1+\gamma\right)+\frac{1}{2}\right)+\frac{x}{2}+1\right)},$$

or:

$$\frac{\frac{\gamma+1}{2}\left(\frac{x}{2}+1\right)^{\gamma}}{\left(\frac{x}{2}+1\right)^{\gamma+1}}=\frac{\gamma+1}{x+2}<\frac{\frac{1}{2}\gamma+\frac{1}{4}x\gamma+\frac{1}{4}x\gamma^2+\frac{3}{8}x^2\gamma-\frac{3}{8}x^2\gamma^2+\frac{1}{2}}{\frac{x}{2}+\frac{1}{2}x\gamma+\frac{1}{8}x^2\gamma+\frac{1}{8}x^3\gamma+\frac{1}{8}x^2\gamma^2-\frac{1}{8}x^3\gamma^2+\frac{1}{2}}$$

The above inequality is equivalent to the following one:

$$\left(\gamma+\frac{1}{2}x\gamma+\frac{1}{2}x\gamma^2+\frac{3}{4}x^2\gamma-\frac{3}{4}x^2\gamma^2+1\right)(x+2)>$$
$$(\gamma+1)\left(x+\gamma x+\frac{1}{4}x^2\gamma+\frac{1}{4}x^3\gamma+\frac{1}{4}x^2\gamma^2-\frac{1}{4}x^3\gamma^2+1\right),$$

or

$$\begin{array}{c}\gamma x+\frac{1}{2}x^2\gamma+\frac{1}{2}x^2\gamma^2+\frac{3}{4}x^3\gamma-\frac{3}{4}x^3\gamma^2+x\\ +2\gamma+x\gamma+x\gamma^2+\frac{3}{2}x^2\gamma-\frac{3}{2}x^2\gamma^2+2\end{array}>$$
$$\begin{array}{c}x+\gamma x+\frac{1}{4}x^2\gamma+\frac{1}{4}x^3\gamma+\frac{1}{4}x^2\gamma^2-\frac{1}{4}x^3\gamma^2+1+\\ \gamma x+\gamma^2x+\frac{1}{4}x^2\gamma^2+\frac{1}{4}x^3\gamma^2+\frac{1}{4}x^2\gamma^3-\frac{1}{4}x^3\gamma^3+\gamma\end{array},$$

or

$$\frac{7}{4}x^2\gamma+\frac{1}{2}x^3\gamma+\gamma+1+\frac{1}{4}x^3\gamma^3>\frac{1}{4}x^2\gamma^3+\frac{3}{4}x^3\gamma^2+\frac{3}{2}x^2\gamma^2.$$

That is

$$\frac{x^2\gamma}{2}\left(\frac{7}{2}-\frac{\gamma^2}{2}-3\gamma\right)+\frac{1}{4}x^3\gamma\left(2+\gamma^2-3\gamma\right)+\gamma+1>0.$$

the above inequality being always true if $\gamma<1$, which completes the proof. ■

Proof of Proposition 3: From (24) and (34) we get:

$$\begin{aligned}&D_c(K,\gamma)-D_r(K,\gamma) \qquad (39)\\ =\;&\left(\frac{1}{2}\right)^{\gamma}\left(\frac{1}{K}\right)^{\gamma-1}-K\left(\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}+\frac{2}{K-1}\frac{1}{\gamma+1}\left(\left(\frac{1}{K+1}\right)^{\gamma}-\left(\frac{1}{2}\right)^{\gamma}\right)\end{aligned}$$

As shown in Propositions 1 and 2, when $\gamma<1$ and N is even, the optimal structure both for the ring and the completely connected structures has all components of size $K+1=2$. As we noticed, when $K=1$ the pattern of exposure is identical for the ring and the completely connected structure, hence the value of the above expression equals zero in that case, as can be verified.[34]

Consider now the case $\gamma>1$, for which $K=N-1$ (i.e. minimal segmentation) is

[34] Strictly speaking, we can show that that its limit for $K\to 1$ equals zero.

optimal for both structures. Evaluating (39) at this value of K we find:

$$D_c(N-1,\gamma) - D_r(N-1,\gamma) =$$
$$= \left[\left(\frac{1}{N}\right)^{\gamma} - \left(\frac{1}{2}\right)^{\gamma}\right]\left[\frac{2}{N-2}\frac{1}{\gamma+1} - \frac{N-1}{1+\gamma}\right] + \left(\frac{1}{2}\right)^{\gamma}\left[\left(\frac{1}{N}\right)^{\gamma-1} - \frac{N-1}{1+\gamma}\right]$$

Since $2 \leq (N-1)(N-2)$ for $N \geq 3$, we have that for all[35] $N > 1 + (1+\gamma)^{\frac{1}{\gamma}}$ the desired conclusion follows:

$$D_c(K,\gamma) - D_r(K,\gamma) < 0.$$

This completes the proof. ■

Proof of Proposition 4: From (24), we can write:

$$D_c(K,\gamma,\gamma',p) = p\,K\,\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,K\,\left(\frac{1}{2K}\right)^{\gamma'}$$

Hence

$$\frac{\partial D_c}{\partial K}(K,\gamma,\gamma',p) = -p\,(\gamma-1)\left(\frac{1}{2K}\right)^{\gamma} - (1-p)\,(\gamma'-1)\left(\frac{1}{2K}\right)^{\gamma'}. \qquad (40)$$

and $\frac{\partial D_c}{\partial K} > 0$ is equivalent to

$$(1-p)\,(1-\gamma')\left(\frac{1}{2K}\right)^{\gamma'} > p\,(\gamma-1)\left(\frac{1}{2K}\right)^{\gamma},$$

or, since $\gamma > 1$ and $\gamma' < 1$,

$$K > \frac{1}{2}\left(\frac{p\,(\gamma-1)}{(1-p)\,(1-\gamma')}\right)^{\frac{1}{\gamma-\gamma'}}.$$

This implies that $D_c(K,\gamma,\gamma',p)$ is minimized at the point

$$\hat{K}(p) = \frac{1}{2}\left(\frac{p\,(\gamma-1)}{(1-p)\,(1-\gamma')}\right)^{\frac{1}{\gamma-\gamma'}}$$

provided this point is admissible, i.e. $\hat{K}(p) \in [1, N-1]$.

[35]Note that $(1+\gamma) < (N-1)^{\gamma-1}\,(N-1) < (N)^{\gamma-1}\,(N-1)$

Compute next the second derivative of $D_c(\cdot)$:

$$\frac{\partial^2 D_c}{\partial K^2}(K,\gamma,\gamma',p) = p\,(\gamma-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,(\gamma'-1)\,\frac{\gamma'}{K}\left(\frac{1}{2K}\right)^{\gamma'}$$
$$\geq p\,(\gamma-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,(\gamma'-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma'}$$
$$= -\frac{\gamma}{K}\frac{\partial D_c}{\partial K}(K,\gamma,\gamma',p)$$

Thus $\frac{\partial^2 D_c}{\partial K^2}(1/2, K, \gamma, \gamma', p) > 0$ for all feasible $K < \hat{K}(p)$, i.e. the function $D_c(\cdot)$ is convex in this range.

The optimal degree of segmentation for the completely connected structure is obtained as a solution of problem 22. Denote by $(K_i^*)_{i=1}^C$ a vector of component sizes that solves this optimization problem. We will show that there exists some appropriate range $[p_0, p_1]$ such that if $p \in [p_0, p_1]$, the optimal component sizes are such that $K_i^* = K_j^* = K^*$ for all $i, j = 1, 2, \ldots, C$ and some common K^* with $2 \leq K^* \leq N-2$.

Choose p_0 such that $\hat{K}(p_0) = \frac{N}{2} - 1$. Such a choice is feasible and unique since by A.3 $N > 4$, $\hat{K}(\cdot)$ is increasing in p, $\hat{K}(0) = 0$, and $\hat{K}(p) \to \infty$ as $p \to 1$. Next we show that, for all $p \geq p_0$, whenever $C \geq 2$,the vector $(K_i^*)_{i=1}^C$ solving problem 22 satisfies:

$$\forall i, j = 1, 2, \ldots, C, \quad K_i^* = K_j^* \leq \hat{K}(p) \tag{41}$$

Let K_i^* and K_j^* stand for any two component sizes that are part of the solution to the optimization problem. First note that, since $\hat{K}(p) \geq N/2 - 1$, if $K_i^* > \hat{K}(p)$ then we must have that $K_j^* < \hat{K}(p)$. But such asymmetric arrangement cannot be part of a solution to problem 22 because $D_c(\cdot, \gamma, \gamma', p)$ is increasing at K_i^* and decreasing at K_j^*. Hence a sufficiently small increase of K_j and a decrease of K_i, which keeps $K_i + K_j$ unchanged, is feasible and allows to decrease the expected mass of defaults. The only possibility, therefore, is that $K_j^* \leq \hat{K}(p)$ and $K_j^* \leq \hat{K}(p)$.

To complete the argument and establish (41), suppose that at an optimum we have $K_i^* \neq K_j^*$ for at least two components i, j. Since, as shown in the previous paragraph, neither K_i^* nor K_j^* can exceed $\hat{K}(p)$, both K_i^*, K_j^* lie in the convex part of the function $D_c(\cdot, \gamma, \gamma', p)$. It follows, therefore, that if we replace these two (dissimilar) components with two components of equal size $\frac{1}{2}(K_i^* + K_j^*)$, feasibility is still satisfied and the overall expected mass of defaults is reduced, contradicting that the two heterogeneous components of size K_i^*, K_j^* belongs to an optimum configuration.

We have thus shown that, when $p \geq p_0$, if at the optimum we have $C \geq 2$, the unique optimal configuration involves a uniform segmentation in components of common size $K^*(p) \leq \hat{K}(p)$. It remains then to show that at the optimum we indeed have $C \geq 2$. At $p = p_0$ the optimum exhibits two components, $C = 2$, since the optimal component size $\hat{K}(p_0) = N/2 - 1$ is feasible. Since $\hat{K}(p)$ is increasing and continuous in p and $D_c(K, \gamma, \gamma', p)$ is continuous in K, by continuity there exists some p_1, with $p_0 < p_1 < 1$, such that for all $p \in (p_0, p_1)$ the expected mass of defaults in a structure with two components, both of size $N/2 - 1$, is still smaller than that in a single component of size N. That is, at the optimum $C \geq 2$.

Since $N/2 - 1 > 1$, this completes the proof that the optimal component size $K^* + 1$ is "intermediate," i.e. satisfies $1 < K^* < N - 1$. ■

Proof of Proposition 5: For the probability distribution of the b shock stated in the claim, the expected mass of firms not directly hit by a b shock who default in a completely connected component of size K when a b shock hits the component is:

$$D_c(K, \gamma, p) = (1 - p)K + pK \left(\frac{1}{2K}\right)^{\gamma}. \tag{42}$$

Differentiating the above expression with respect to K yields:

$$\frac{\partial D_c(K, \gamma, p)}{\partial K} = (1 - p) - (\gamma - 1)\, p \left(\frac{1}{2K}\right)^{\gamma},$$

which is negative for all K as long as (26) is satisfied. This establishes that the optimal degree of segmentation for the complete structure is minimal, that is obtains at $K = N - 1$.

Next, using (34) and (20), noting that $\bar{L} > \frac{1}{H} = K + 1$ for all K, we obtain the following expression for the expected mass of defaults in the case of the ring structure:

$$D_r(K,\gamma,p) = (1-p)\left(K - \left(K - \frac{2}{K+1}\right)\frac{K+1}{\bar{L}}\right)$$
$$+\,p\left[\left(\frac{K}{\gamma+1} - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}\right] + p\left[\frac{1}{K-1}\frac{1}{2^{\gamma-1}}\frac{1}{\gamma+1}\right]$$

It suffices then to show that the expected mass of defaults is smaller for the ring than for the completely connected structure when $K = N-1$: $D_c(N-1,\gamma,p) > D_r(N-1,\gamma,p)$ or, substituting the above expressions:

$$(1-p)\,(N-1) + p\,(N-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma} > (1-p)\left(N-1-\tfrac{N^2-N-2}{2N-1}\right)+$$
$$p\left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + p\left[\tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1}\right],$$

which can be rewritten as

$$\left(\tfrac{1-p}{p}\right)\tfrac{N^2-N-2}{2N-1} >$$
$$\left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + \tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1} - (N-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma}.$$

Using (26) the above inequality holds for an open interval of values of p if

$$(\gamma-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma}\tfrac{N^2-N-2}{2N-1} >$$
$$\left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + \tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1} - (N-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma}$$

or

$$(\gamma-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma}2\left(\tfrac{(N-1)^2+N-3}{2(2(N-1)+1)}1\right) + (N-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma}$$
$$-2\left(\tfrac{(N-1)}{2}\left(\tfrac{1}{\gamma+1}\right) - \tfrac{1}{\gamma+1}\left(\tfrac{1}{N-2}\right)\right)\left(\tfrac{1}{N}\right)^{\gamma} - \left(\tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{(\gamma+1)}\right) > 0$$

Noticing that by A.3 and (25) we have $N \geq 5$ and this in turn implies

$$\frac{(N-1)^2+N-3}{4\,(N-1)+2} \geq \frac{N-1}{4},$$

a sufficient condition for the above inequality to hold is that:

$$
\begin{aligned}
& (\gamma-1)\ \left(\frac{1}{2\,(N-1)}\right)^{\gamma} 2\left(\frac{N-1}{4}\right)+\frac{2}{\gamma+1}\frac{1}{N-2}\left(\frac{1}{N}\right)^{\gamma} \\
& +(N-1)\ \left(\frac{1}{2\,(N-1)}\right)^{\gamma}-\left(\frac{N-1}{\gamma+1}\right)\left(\frac{1}{N-1}\right)^{\gamma}-\left(\frac{1}{N-2}\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) \\
= \ & \frac{N-2}{(N-1)^{\gamma-1}}\left((\gamma-1)\ \frac{1}{2^{\gamma+1}}+\frac{1}{2^{\gamma}}-\frac{1}{\gamma+1}\right)+\frac{2}{\gamma+1}\left(\frac{1}{N}\right)^{\gamma}-\left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right)>0.
\end{aligned}
$$

Since $\gamma \in (1,2)$, this inequality is in turn satisfied if the following hold:

$$
\left[\frac{N-2}{(N-1)^{\gamma-1}}+\left(\frac{1}{N}\right)^{\gamma}\right]\left(\frac{\gamma+1}{2^{\gamma+1}}-\frac{1}{\gamma+1}\right)-\left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right)>0
$$

or

$$
(N-1)^{2-\gamma}-\left(\frac{1}{(N-1)^{\gamma-1}}-\frac{1}{N^{\gamma}}\right)>\frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}}-\frac{1}{\gamma+1}}
$$

which is implied by the inequality

$$
(N-1)^{2-\gamma}-\left(\frac{1}{4^{\gamma-1}}-\frac{1}{5^{\gamma}}\right)>\frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}}-\frac{1}{\gamma+1}}
$$

that is in turn equivalent to (25). This completes the proof of the proposition. ■

Further details of the proof of Proposition 9:

Noting that $a_{11}^s - a_{ii}^s = (1-\theta)^2(\beta-1)/\beta$, the exposure matrix for the star structures can be conveniently rewritten as follows:

$$
A^s = \begin{pmatrix}
a_{11}^s & (1-a_{11}^s)/\beta & (1-a_{11}^s)/\beta & \cdots & (1-a_{11}^s)/\beta \\
1-a_{11}^s & a_{ii}^s & (a_{11}^s-a_{ii}^s)/(\beta-1) & \cdots & (a_{11}^s-a_{ii}^s)/(\beta-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1-a_{11}^s & (a_{11}^s-a_{ii}^s)/(\beta-1) & (a_{11}^s-a_{ii}^s)/(\beta-1) & \cdots & a_{ii}^s
\end{pmatrix},
$$

Recall that $a_{ii}^s = 1/2$ while a_{11}^s is determined, together with θ, by (30) and (31). Its properties are characterized below:

LEMMA 4 *For all $\beta > 2$, the solution of (30) and (31) is unique and given by continuous, monotonically increasing functions $\theta(\beta)$ and $a^s_{11}(\beta)$, such that*

$$5/9 \leq a^s_{11}(\beta) < 2 - \sqrt[2]{2} \tag{43}$$

Proof of Lemma 4:

It can be easily verified that for all $\beta > 2$ there is only one admissible (i.e., lying between 0 and 1) solution of (30), given by

$$\frac{2 + \sqrt{2\beta^2 - 2\beta}}{2\beta + 2}.$$

This expression defines the function $\theta(\beta)$, which is increasing if and only if the following inequality is satisfied:

$$\frac{(4\beta - 2)}{2\sqrt{2\beta^2 - 2\beta}}(2\beta + 2) > \left(4 + 2\sqrt{2\beta^2 - 2\beta}\right),$$

which is equivalent to

$$2\beta^2 + \beta - 1 > 2\sqrt{2\beta^2 - 2\beta} + 2\beta^2 - 2\beta$$

or

$$9\beta^2 - 6\beta + 1 > 8\beta^2 - 8\beta$$

always satisfied for $\beta > 2$. The minimal value of θ in this range is then $\theta(2) = 2/3$, while the maximum is $\lim_{\beta\to\infty} \theta(\beta) = 1/\sqrt[2]{2}$.

Also, $a^s_{11}(\beta)$ is also increasing in β

$$\frac{da^s_{11}}{d\beta} = 2(2\theta - 1)\frac{d\theta}{d\beta}.$$

Hence its minimum value is $a^s_{11}(\beta) = 5/9$ and its maximum is $2 - \sqrt[2]{2}$. ■

The precise expression of $G_{star}(l) = \left(G_{star}(l; s) + G_{star}(l; \ell)\right)/2$ is obtained from that of $G_{star}(l; s)$ and $G_{star}(l; \ell)$ and is given by:

$$G_{star}(l) = \begin{cases} 0 & \text{for } l \leq 2 \\ \frac{1}{2} & \text{for } 2 < l \leq \frac{1}{1-a_{11}^s} \\ \frac{1}{2} + \frac{1}{2}\beta & \text{for } \frac{1}{1-a_{11}^s} < l \leq \frac{\beta}{a_{11}^s} \\ \frac{1}{2} + \frac{1}{2}2\beta & \text{for } \frac{\beta}{a_{11}^s} < l \leq \min\left\{\frac{\beta-1}{a_{11}^s-1/2}, \frac{\beta^2}{1-a_{11}^s}\right\} \\ \begin{array}{c} \frac{\beta}{2} + \beta \text{ if } \frac{\beta-1}{a_{11}^s-1/2} \leq \frac{\beta^2}{1-a_{11}^s} \\ \frac{1+\beta}{2} + \beta \text{ otherwise} \end{array} & \text{for } \min\left\{\frac{\beta-1}{a_{11}^s-1/2}, \frac{\beta^2}{1-a_{11}^s}\right\} < l \leq \max\left\{\frac{\beta-1}{a_{11}^s-1/2}, \frac{\beta^2}{1-a_{11}^s}\right\} \\ 2\beta & \text{for } l > \max\left\{\frac{\beta-1}{a_{11}^s-1/2}, \frac{\beta^2}{1-a_{11}^s}\right\} \end{cases}, \tag{44}$$

since again it can be verified, given the previous lemma, that

$$2 < \frac{1}{1-a_{11}^s} < \frac{\beta}{a_{11}^s} < \min\left\{\frac{\beta-1}{a_{11}^s - 1/2}, \frac{\beta^2}{1-a_{11}^s}\right\}.$$

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