Testing Chaotic Dynamics via Lyapunov Exponents

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ABSTRACT

In this paper, we propose a new test, based on the stability of the largest Lyapunov exponent from different sample sizes, to detect chaotic dynamics in economic and financial time series.

We apply this new test to the simulated data used in the single-blind controlled competition among tests for nonlinearity and chaos provided by Barnett *et al.* (1997), both for small samples (380 observations) and for large samples (2000 observations). The results suggest that the new test has high power against different stochastic alternatives (both linear and nonlinear) and that behaves well in small samples.

JEL classification numbers: C13, C14, C15, C22

KEY WORDS: Chaos, Nonlinear Dynamics, Bootstrapping

1. Introduction

In a dissipative dynamical system, the existence of a positive Lyapunov exponent is taken generally as an indication that the system is chaotic. Lyapunov exponents provide information on the intrinsic instability of the trajectories of the system, and are computed as the average rate of exponential convergence or divergence of nearby trajectories in the phase space.

In recent years, there has been a burgeoning literature on the calculation of Lyapunov exponents of an unknown dynamical system reconstructed from a single time series. Wolf *et al.* (1985)'s seminal paper provides an algorithm to compute Lyapunov exponents in empirical applications, but it is sensitive to both the number of observations and the degree of noise in the data. More recently, however, some papers have proposed new methods of estimating Lyapunov exponents with good performance even for small samples [see, among others, Dechert and Gençay (1992), Abarbanel *et al.* (1991,1992), and Rosentein *et al.* (1993)].

There are many papers using Lyapunov exponents to detect chaotic dynamics in financial time series, especially in exchange rate series. Earlier examples of research in this area are those of Bajo-Rubio *et al.* (1992) and Dechert and Gençay (1992). In these papers Lyapunov exponents are used to distinguish between linear deterministic processes (with negative Lyapunov exponents) and nonlinear, chaotic deterministic processes (where the largest Lyapunov exponents is positive). These and other papers have been criticised for the absence of a distributional theory providing a statistical framework for hypothesis testing using the calculated Lyapunov exponents. However, Gençay (1996) presents a methodology to compute the empirical distributions of Lyapunov exponents using a blockwise bootstrap technique. This methodology provides a formal test of the largest Lyapunov exponent equals some hypothesised value, and can be used to test for chaotic dynamics. Gençay (1996)'s test is particularly useful in those cases where the largest Lyapunov exponent is positive, but very closed to zero. Later, Bask and Gençay (1998) utilise the same statistical framework to propose a test for the presence of a positive Lyapunov exponent in an observed time series. The numerical examples show that both Gençay (1996) and Bask and Gençay (1998) test statistics behave well in small samples. Finally, Bask (1998) using Bask and Gençay (1998)'s test finds evidence that some exchange rates can be characterised by deterministic chaos.

Despite the growing interest in the econometric literature the distinction between non-linear deterministic processes and non-linear stochastic processes, much disagreement and controversy has arisen about the available results. A key paper in this area is that of Barnett *et al.* (1997), where data series were simulated

from different generating models in order to evaluate the behaviour, both for large (2000 observations) and small (380 observations) samples, of five highly regarded tests for nonlinearity or chaos. The tests considered in that paper are the Hinich (1982)'s bispectral test, the BDS test (Brock *et al.*, 1996), the Nychka *et al.* (1992)'s Lyapunov exponent test, the White (1989)'s test and the Kaplan test. The results of that experiment provided much surprising information about the power function of some of such tests. Barnett *et al.* (1997) conclude that none of these tests has the ability to isolate the origins of the nonlinearity or chaos to be in the structure of the economy.

The aim of this paper is to propose a new test for the presence of chaos, based on the behaviour of the estimated Lyapunov exponents with different sample sizes. The new test shows strong power against stochastic processes hence improving those tests proposed by Gençay (1996) and Bask and Gençay (1998).

The paper rest of the paper is organised as follows. Section 2 presents the statistical framework that we use in this paper. Section 3 discusses the stability of largest Lyapunov exponent with sample size. Section 4 proposes the new test for distinguish chaos from random behaviour. Section 5 reports the results of applying our test to the simulated data used in the Barnett (1997)'s single-blind controlled competition, with especial emphasis in the Feigenbaum series. Finally, Section 6 provides some concluding remarks.

2. Lyapunov exponent and the ergodic theory for dynamical systems

In order to examine the properties of deterministic dynamical system we make use of the ergodic theory, since it provides a statistical framework to distinguish different degrees of complexity of attractors and motions [see Eckmann and Ruelle (1985) for a survey]. The ergodic theory allows us to describe the time averages of a dynamical system and to consider that transients become irrelevant: once transients are over the motion of the dynamical system settles typically near a subset of $\mathfrak{R}^n$, called an attractor. In the particular case of dissipative systems, where the phase-space volumes are concentrated by the time evolution, the volume occupied by the attractor is in generally very small in relation to the phase space. Even if a system contracts volume, it does not mean that it contracts length in all directions: some directions may be stretched and some directions contracted. It implies that, even in a dissipative system, the final motions may be unstable within the attractor. This instability usually manifests itself in *sensitive dependence on initial conditions* what means an exponential separation of orbits, as time goes on, of points which initially were very close each other on the attractor. In this case, we say that the attractor is a *strange attractor* and that the system is *chaotic*.

Statistical averages can be computed either in terms of time averages or space averages. Let us consider, for simplicity, a discrete dynamical system of dimension n $\bar{x}_{t+1} = \bar{F}(\bar{x}_t)$, where $\bar{F} : \mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ is a vectorial differentiable function. The *time average* of a function φ along a (forward) trajectory $\bar{x}_i$ with initial condition $\bar{x}_0$, of a discrete dynamical system is defined by

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N-1} \varphi(\bar{x}_i)$$

In a similar way for a continuous flow ϕ_t , arising from a continuous dynamical system $\frac{d\bar{x}}{dt} = \bar{F}(\bar{x})$ the time average of a function along a (forward) trajectory is

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(\phi_t(x)) dt$$

The time averages often depends on initial conditions. Nevertheless, when the dynamical system has an attractor, all trajectories have the same statistical properties.

In order to compute space averages it becomes useful to define a probability measure. The weight with which the space average has to be taken is an *invariant measure* (i.e., a measure that does not change under the action of the dynamics of the system). In other words, if the probability associated with a given set is equal to the probability of the sets that are mapped into it. More precisely, a measure μ is invariant under the map $\bar{F}$ if, for any subset S of points in $\mathfrak{R}^n$ in the support of μ ,

$$\mu(S) = \mu(\bar{F}^{-t}(S)), \quad t > 0$$

A basic property of the ergodic theory is that it allows us to consider only the long-run behavior of a system, circumventing the need of specify the transitory states. Therefore, if the long-run behavior of the system is on an attractor, even though the geometric study of attractors presents great mathematical difficulties, the ergodic theory allows us to simplify the problem, shifting attention from attractors to statistic in the phase space throw invariant measures.

An invariant probability measure μ is *indecomposable* or *ergodic* if may not be decomposed into several different pieces, each of them being again invariant (that is, the system has only one attractor). If μ is a ergodic measure on an attractor $A \subset \mathfrak{R}^n$ the *ergodic theorem* asserts that for every continuous function φ , and for almost initial conditions, time average equals a space average, that is

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N-1} \varphi(\bar{x}_i) = \int_A \varphi \, d\mu,$$

in a discrete dynamical system, and

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(\phi_t(\bar{x})) dt = \int_A \varphi \, d\mu$$

in a continuous dynamical system.

This in turn implies that the fraction of time that a dynamical system is situated in a region R of the phase space will be equal to the fraction of the area of R in the total area of the phase space. This result makes it possible to introduce probability distributions that are invariant to the dynamics of the system. Therefore, the ergodic theory allows us to elaborate a statistic framework for the dynamical systems capable to distinguish between different degrees of dynamic complexity.

The time averages of typical trajectories are described by the *natural measure*, that is a measure that describes time averages for a set of initial conditions of positive Lebesgue measure in the phase space. A dynamical system may have many natural measures, associated with different families of trajectories. For instance, in a dissipative dynamical system, each attractor has its own natural measure. Each measure allows us to obtain the probability of finding the system on a given region of the attractor. By the ergodic theorem, the natural measure can be estimated numerically in terms of the frequency with which trajectories visit different parts of the attractor, for example by sampling uniform time intervals and using a histogram. The natural density associated to the natural measure is then given by:

$$\rho(\vec{x}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \delta^d(\vec{x} - \vec{x}_t)$$

where δ^d is the Dirac delta function.

A measure of complexity in chaotic motion may be obtain by analyzing the sensitivity of the dynamical behavior to initial conditions given by two infinitely close initial states. For chaotic systems nearby points in the phase space separate exponentially with time. Let us illustrate the basic idea with a discrete dynamical system of dimension n , $\vec{x}_{t+1} = \vec{F}(\vec{x}_t)$. In order to examine the stability of the trajectories of the system, let us consider how the system amplifies a small difference between the initial conditions $\vec{x}_0$ and $\vec{x}'_0$:

$$\vec{x}_T - \vec{x}'_T = \vec{F}^T(\vec{x}_0) - \vec{F}^T(\vec{x}'_0) \cong D\vec{F}^T(\vec{x}_0)(\vec{x}_0 - \vec{x}'_0)$$

where $\vec{F}^T(\vec{x}_0) = \vec{F}(\vec{F}(\dots\vec{F}(\vec{x}_0)\dots))$ denotes the T successive iterations of the dynamical system starting from the initial condition $\vec{x}_0$, and where $D\vec{F}^T(\vec{x}_0)$ is the Jacobian of function $\vec{F}(\vec{x})$.

By the rule of the chain, we have

$$D\vec{F}^T(\vec{x}_0) = D\vec{F}(\vec{x}_{T-1}) D\vec{F}(\vec{x}_{T-2}) \dots D\vec{F}(\vec{x}_0)$$

In this context, the Lyapunov exponents are defined as follows (Guckenheimer and Holes, 1990): Let us consider the family of subspaces $V_i^{(1)} \supset V_i^{(2)} \supset \dots \supset V_i^{(n)}$ in the tangent space at $\vec{F}^i(\vec{x})$ and the numbers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ with the properties that:

- (1) $D\vec{F}(V_i^{(j)}) = V_{i+1}^{(j)}$
- (2) $\dim V_i^{(j)} = n + 1 - j$
- (3) $\lim_{T \rightarrow \infty} \frac{1}{T} \ln || \sqrt{(D\vec{F}^T)^* \cdot (D\vec{F}^T)}(\vec{x}_0) || = \lambda_j$ for all $\vec{x}_0 \in V_0^{(j)} - V_0^{(j-1)}$,

where $(D\vec{F}^T)^*$ is the transpose of $D\vec{F}^T$

Then, the real numbers λ_j are called the *Lyapunov exponents* of $\vec{F}$ at $\vec{x}_0$. Lyapunov exponents offer information on how orbits on the attractor move apart (or together) under evolution of the dynamics. One can also define them by the rate of stretching or shrinkage of line segments, areas, and various dimensional subvolumes in the phase space. Line segments grow or shrink as $e^{t\lambda_1}$, areas as $e^{t(\lambda_1+\lambda_2)}$ and so forth. If one or more of the Lyapunov exponents are positive, then we have *chaos* in the motion of the system. The sum of the Lyapunov exponents is negative ($\lambda_1 + \lambda_2 + \dots + \lambda_n < 0$) for dissipative systems [see Abarbanel (1996)].

The possibility of obtaining, in a deterministic dynamical system, Lyapunov exponents that are representative of short-run divergences in trajectories with very closed initial points is based on Oseledec (1968)'s *multiplicative ergodic theorem*. If we assume that there exists an ergodic measure of the system, this theorem justifies the use of arbitrary phase space directions when calculating the largest Lyapunov exponent. The Lyapunov exponents have then a global sense, allowing to characterize the complexity of a deterministic dynamical system of dimension n simply by n real numbers.

Oseledec (1968)'s multiplicative ergodic theorem states that, under wide general conditions for function $\vec{F}$, the limit in expression (3) does exist for almost all $\vec{x}_0$ (with respect to the invariant measure μ) and is independent of the initial condition $\vec{x}_0$ considered (except for a set of null measure). Therefore, the multiplicative ergodic theorem implies that the Lyapunov exponents are invariant numbers representing "globally" the complexity of the dynamical system under study, independently of the initial condition considered.

As can be seen, Oseledec theorem is based on the ergodic theory of deterministic dynamical systems and justifies the use of arbitrary phase space directions when calculating the largest Lyapunov exponents. Nevertheless, as Whang and Linton (1999) and Tong (1990) point out, Lyapunov exponents can be interpreted within the standard nonlinear time series frame work as a measure of local stability and is of interest even outside of any direct connection with deterministic chaos.

Our simulations will show an essential difference between chaotic and stochastic processes via Lyapunov exponents. If we want reconstruct trajectories of a time series in a phase space that are sampled from a stochastic process, there is not guarantee of convergence in any algorithm towards the largest Lyapunov exponent, because the Lyapunov exponents are not necessarily stable and independent of initial conditions and sample size. For stochastic processes, the algorithm is only capable to estimate *local Lyapunov exponents*. Local Lyapunov exponents are a measure of local stability of the process, and may be highly dependent on the initial condition considered and, as we will show, on the sample size.

3. Stability of largest Lyapunov exponents with sample size

3.1 The Rosenstein *et al.* (1993) method for estimating the largest Lyapunov exponents

There are several suitable estimation methods in order to obtain Lyapunov exponents based on kernels, nearest neighbours, splines, local polynomials and neural nets [see Härdle and Linton (1994) for a general discussion]. McCaffrey *et al.* (1992) distinguish two classes of methods for estimating the largest Lyapunov exponent λ_{max} : (i) Direct methods like Wolf *et al.* (1985) that assume that the initial divergence $(\bar{x}_0 - \bar{x}'_0)$ grows at exponential rate given by λ_{max} ; and (ii) Jacobian methods, where data are used to estimate the Jacobians, being λ_{max} calculated from the estimated Jacobians, like proposed by MacCaffrey *et al.* (1992) or Gençay (1996).

In this paper we use a simple direct method for estimating the largest Lyapunov exponent λ_{max} proposed by Rosenstein *et al.* (1993). Let us consider an observed time series $\{x_1, \dots, x_N\}$. Following Takens (1981) theorem, we start by reconstructing the phase-space vector $\{x_1^d, \dots, x_{N-d+1}^d\}$, where $x_t^d = (x_t, \dots, x_{t+d-1})$ and $t \leq N - d + 1$. For each point x_t^d , we search for the nearest neighbour point $x_{t^*}^d$ in the reconstructed phase space that minimises the distance to that reference point:

$$d_t(0) = \min_{t^*} \|x_t^d - x_{t^*}^d\|,$$

where $\| \cdot \|$ denotes the Euclidean norm. To consider each pair of neighbours as nearby initial conditions for different trajectories, the temporal separation between them should be greater than the mean period of the time series:

$$|t - t^*| > \text{mean period}$$

This mean period can be estimated as the reciprocal of the mean frequency of the power spectrum of the time series under study.

The divergence between the nearest neighbours x_t^d and $x_{t^*}^d$ takes place at a rate approximated by the largest Lyapunov exponent:

$$d_t(i) \cong d_t(0) \exp(\lambda_{\max} i) \quad \text{para } i = 1, \dots, N$$

where i is the number of discrete-time steps following nearest neighbour. Taking the logarithm in both sides of this last expression, we obtain:

$$\log(d_t(i)) \cong \log(d_t(0)) + \lambda_{\max} i$$

For each value t between 1 and $N-d+1$, this equation represents a set of approximately parallel lines, each with a slope approximately proportional to $\lambda_{\max}$. The largest Lyapunov exponent is then estimated using a least-square fit with a constant to the average line defined by $\langle \log(d_t(i)) \rangle$, where $\langle \rangle$ denotes the average value over all values of t .

In the method proposed by Rosenstein *et al.* (1993), there are two key parameters to estimate the largest Lyapunov exponent: the embedding dimension (d) and the number of discrete-time steps (i) allowed for divergence between nearest neighbours x_t^d and x_{t+i}^d in the phase space.

As shown in Rosenstein *et al.* (1993) the value for the largest Lyapunov exponent can change substantially with these two parameters.

3.2 Moving blocks bootstrapping

Gençay (1996) proposed a statistical framework for testing chaotic dynamics using a moving blocks bootstrap procedure.

Consider a sequence $\{X_1, \dots, X_N\}$ of weakly dependent stationary random variables, being $\{x_1, \dots, x_N\}$ a time series realisation of such a stochastic process. According to Künsch (1989) and Liu and Singh (1992), the distribution of certain estimators of interest can be consistently constructed by applying moving blockwise bootstrap. Let $B_t^d = \{x_t, \dots, x_{t+d-1}\}$ denote a moving block of d consecutive observations. For a time series of N elements, we can form a set $\{B_1^d, \dots, B_{N-d+1}^d\}$ of blocks with length d . Let us consider $k = \text{int}(N/d)$ [where $\text{int}()$ denotes the integer part], by resampling with replacement of k blocks denoted by $\{B_{i_1}^d, \dots, B_{i_k}^d\}$, we will form the bootstrap sample.

In order to obtain the sample distribution of the largest Lyapunov exponent $\lambda_{\max}$, we will repeat this procedure to construct a sequence of sub-families of k blocks taken with replacement from the family of d -dimensional blocks

$\{B_1^d, \dots, B_{N-d+1}^d\}$, that can be generated with the time series $\{x_1, \dots, x_T\}$. For each of subfamily of k blocks, we can apply some standard procedure to compute for the largest Lyapunov exponent $\tilde{\lambda}_{max}$ taking the pairs of nearest neighbours from each subfamily of blocks. Repeating this process a large number of times, we will obtain the empirical distribution of the largest Lyapunov exponent $\tilde{\lambda}_{max}$.

Gençay (1996) used this methodology to test if the largest Lyapunov exponent λ_{max} estimated from a time series is equal to some value λ_1 . His test is as follows:

$$H_0: \lambda_{max} = \lambda_1,$$

$$H_1: \lambda_{max} \neq \lambda_1.$$

To implement this test, he obtain the empirical distribution of λ_{max} and calculate the quantiles $q(2.5\%)$ and $q(97.5\%)$ at 2.5% and 97.5%, being $Pr(\lambda_{max} < q(2.5)) = 0.025$ and $Pr(\lambda_{max} < q(97.5)) = 0.975$. If, on the one hand, $\lambda_1 \in [q(2.5\%), q(97.5\%)]$, the null hypothesis cannot be rejected. If, on the other hand, $\lambda_1 \notin [q(2.5\%), q(97.5\%)]$, the null hypothesis is rejected.

This test was improved by Bask and Gençay (1998) in order to transform it into a test of positive largest Lyapunov exponent, and therefore a test for chaotic dynamics in a time series. The null and the alternative hypotheses are as follows:

$$H_0: \lambda_{max} = 0,$$

$$H_1: \lambda_{max} > 0$$

The test scheme consists of the following steps:

- i.) Reconstruct the phase space of the time series $\{x_1, \dots, x_N\}$ with a embedding dimension d and estimate the largest Lyapunov exponent $\hat{\lambda}_{max}$ using any existing algorithm. Each d -history of the reconstructed phase space will be considered as a block, obtaining in this way a sequence of blocks $\{B_1^d, \dots, B_{N-d+1}^d\}$.
- ii.) Resample, with replacement, k blocks of the reconstructed phase space, being $k = \text{int}(N/d)$. The subfamily of blocks $\{B_{i_1}^d, \dots, B_{i_k}^d\}$ constitutes the bootstrap sample.
- iii.) From this subfamily, estimate the largest Lyapunov exponent $\tilde{\lambda}_{max}$ for the time series under study and calculate $\tilde{\lambda}_{max} - \hat{\lambda}_{max}$.

- iv.) Repeat steps ii)-iii) a large number of times to construct an empirical distribution of $\tilde{\lambda}_{max} - \hat{\lambda}_{max}$
- v.) Construct a one-sided 97.5% confidence interval by calculating the critical value as $\hat{\lambda}_{max} - q(97.5\%)$, following from $Pr\{\hat{\lambda}_{max} - \lambda_{max} < q(97.5\%)\} = 0.975$, where $q(97.5\%)$ is the quantile for the distribution in step iv)
- vi.) If $\hat{\lambda}_{max} - q(97.5\%) > 0$, then the null hypothesis is rejected, which means that the dynamics is chaotic.

3.3 The largest Lyapunov exponent in small samples

In the theory of dynamical systems, a chaotic system is characterised by globally bounded trajectories in the phase space with positive largest Lyapunov exponent, while, in theory, a white noise process has an infinite largest Lyapunov exponent [see Schuster, (1988)].

Nevertheless, in practical implementations, using finite time series, any standard algorithm for calculating the largest Lyapunov exponent will find a finite, positive value for this exponent for a white noise process. Therefore, the largest Lyapunov exponent on its own is not able to distinguish between a chaotic, non-linear deterministic process and a white noise process. This problem is especially relevant in financial time series, where non-linear stochastic processes, such as GARCH processes, are usually postulated as alternative models to the chaotic behaviour [see, e. g., Hsieh (1991)].

Following Barnett et al. (1997), let us consider samples of size 380 and 2000 observations of the following five models:

Model I:

Model I is a fully deterministic, chaotic Feigenbaum recursion of the form:

$$y_t = 3.57 y_{t-1} (1 - y_{t-1}),$$

where the initial condition was set at $y_0 = 0.7$.

Model II:

Model II is a GARCH process of the following form:

$$y_t = h_t^{1/2} u_t$$

where h_t is defined by

$$h_t = 1 + 0.1 y_{t-1}^2 + 0.8 h_{t-1}, \text{ with } h_0 = 1 \text{ and } y_0 = 0.$$

Model III:

Model III is a nonlinear moving average (NLMA) process:

$$y_t = u_t + 0.8u_{t-1}u_{t-2}.$$

Model IV:

Model IV is an ARCH process of the following form:

$$y_t = (1 + 0.5y_{t-1}^2)^{1/2} u_t,$$

with the value of the initial observations set at $y_0 = 0$, and

Model V:

Model V is an ARMA model of the form:

$$y_t = 0.8y_{t-1} + 0.15y_{t-2} + u_t + 0.3u_{t-1},$$

with $y_0 = 1$ and $y_1 = 0.7$.

With the four stochastic models, the white noise disturbances, u_t , are sampled independently from a standard normal distribution. Note that only Model I is chaotic.

In Figure 1, we report the results of applying Rosenstein *et al.* (1993)'s algorithm in order to calculate largest Lyapunov exponents of series of sizes 380 and 2000, used in Barnett *et al.* (1997)'s competition. Calculations in the Rosenstein *et al.* (1993)'s algorithm were performed with an embedding dimension $d=3$ and a number of discrete-time steps allowed for divergence between nearest neighbours $i=2$.

Given the evidence presented in Figure 1, the existence of a positive largest Lyapunov exponent does not allow, on its own, to infer the presence of chaos in a given time series. If we look for the largest Lyapunov exponent in the statistical framework proposed by Bask and Gençay (1998), we will also conclude that the largest Lyapunov exponent is positive in all five cases examined.

However, Figure 1 shows an interesting and essential difference between chaotic and stochastic processes. While the largest Lyapunov exponent in the case of the Feigenbaum map is practically invariant when increasing the sample size, in the cases of all stochastic processes, the largest Lyapunov exponent increases with the sample size. This behaviour remembers the well-known process of saturation,

in a chaotic time series, of the correlation dimension when the embedding dimension increases. In fact this is the base of the test proposed by Grassberger and Procaccia (1983) in order to detect deterministic chaos.

The reason for stability of largest Lyapunov exponent with the sample size can be found in Oseledec (1968)'s theorem. This theorem assures, for chaotic time series, the possibility of make short-run forecast based on the reconstructed phase space. The Lyapunov exponents are nothing but a measure (in exponential scale) of the mean forecast errors using the nearest neighbour points in the phase space. However, when analysing a time series generated by a non-deterministic stochastic process nothing guaranties the stability of Lyapunov exponents. Oseledec (1968)'s theorem only affects deterministic processes via ergodic theory. As the number of observations in the series increases, the variability of the largest Lyapunov exponent will be greater and, therefore, the largest Lyapunov exponent will also increase unlimited with the sample size.

4. A New test for distinguish chaos from random behaviour via Lyapunov exponents

In this section, we propose a new test, based on the distribution of the largest Lyapunov exponent from different sample sizes, to detect chaotic dynamics in economic and financial time series. As we will see, this new test has high power against different stochastic alternatives, both linear and nonlinear.

Let be a time series of length N , $\{x_1, x_2, \dots, x_N\}$. Let us divide the series in different subsamples $\{x_1, x_2, \dots, x_{T_1}, \dots, x_{T_2}, \dots, x_{T_{r-1}}, \dots, x_{T_r=N}\}$, and let us consider an empirical distribution of the largest Lyapunov exponent from 100 moving block bootstrap of such time series for the different subsamples $\{x_1, x_2, \dots, x_{T_i}\}$, for $i=1, \dots, r$. Let $\langle \lambda_{\max}(T_i) \rangle$ be the mean of distributions of 100 largest Lyapunov exponents calculated from those sample sizes. Given that we have shown that the largest Lyapunov exponent is invariant when increasing the sample size in a chaotic series, but it increases with the sample size in a stochastic process, we propose to use $\langle \lambda_{\max}(T_i) \rangle$ to test for the stability of the largest Lyapunov exponent. To that end, we can use several test statistics: a parametric t-test, a non-parametrical Kruskal-Wallis test, and a classical regression test. In all of three versions of the test the null hypothesis is deterministic chaos and the alternative hypothesis is any stochastic process.

4.1. Welch Parametric t-test

First of all we may apply the traditional parametric t test to compare the mean of two populations with unknown variance, assuming that for each sample size the largest Lyapunov exponent follows a Normal distribution. From a theoretical point of view, it is admissible to say that the use of blockwise bootstrap techniques to calculate the largest Lyapunov exponent leads, via the central limit theorem, to a normal distributions of that exponents.

The statistical test is as follows:

$$H_0 : \langle \lambda_{\max}(T_i) \rangle = \langle \lambda_{\max}(T_j) \rangle \text{ for all } i \neq j, \text{ chaotic behaviour}$$

$$H_1 : \langle \lambda_{\max}(T_i) \rangle \neq \langle \lambda_{\max}(T_j) \rangle \text{ for all } i \neq j, \text{ stochastic behaviour}$$

Therefore, assuming normal distributions for the populations of largest Lyapunov exponents, we compare the population means with the statistics:

$$t' = \frac{\mu_x - \mu_y}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}}$$

where μ_x and μ_y are the sampling means and s_x^2 and s_y^2 are the sampling variances.

Welch shows (see Kendall and Stuart, 1969) that the statistic t follows, approximately, a t-distribution with ν degrees of freedom, where

$$\frac{1}{\nu} = \frac{1}{n-1} \left[\frac{\frac{s_x^2}{n}}{\frac{s_x^2}{n} + \frac{s_y^2}{m}} \right]^2 + \frac{1}{m-1} \left[\frac{\frac{s_y^2}{m}}{\frac{s_x^2}{n} + \frac{s_y^2}{m}} \right]^2$$

converges to a normal distribution for large sample sizes.

4.2. Non-parametrical Kruskal-Wallis test

It is possible improve our test transforming it in a non-parametric test and considering as null hypothesis the equity of the mean for several populations of largest Lyapunov exponents for all sample sizes, $T_1, T_2, \dots, T_k$.

The statistical test is now

$$H_0 : \langle \lambda_{\max}(T_1) \rangle = \langle \lambda_{\max}(T_2) \rangle = \dots = \langle \lambda_{\max}(T_n) \rangle, \text{ chaotic behaviour}$$

$$H_1 : \langle \lambda_{\max}(T_1) \rangle \neq \langle \lambda_{\max}(T_2) \rangle \neq \dots \neq \langle \lambda_{\max}(T_n) \rangle, \text{ stochastic behaviour}$$

In this case we use a more elaborate criterion, known in the non-parametric statistical literature as Kruskal-Wallis test [see, for instance, Noether (1991)].

The basic idea in this test is to take together the largest Lyapunov exponents for all the sample sizes and replace the original observations by ranks, that is, replace the smallest exponent by 1, the next smallest by 2, and so on, using mid-ranks where ties occur among the original scores of exponents.

Let us assume that there are n_i largest Lyapunov exponents for the sample size T_i , and the total number of observations is $N = n_1 + n_2 + \dots + n_k$. Let R_i be the sum of the ranks for the exponents with sample size T_i . Kruskal and Wallis consider the statistic

$$H = \frac{12}{N(N+1)} \left[\frac{1}{n_1} \left(R_1 - n_1 \frac{N+1}{2} \right)^2 + \dots + \frac{1}{n_k} \left(R_k - n_k \frac{N+1}{2} \right)^2 \right] - 3(N+1)$$

The formula for H is particularly simple when $n_1 = \dots = n_k = n$, as in our case.

$$H = \frac{12}{nN(N+1)} \left[R_1^2 + \dots + R_k^2 \right] - 3(N+1)$$

With the null hypothesis that all maximum Lyapunov exponents are sampled of the same continuous distribution, Kruskal and Wallis show that the statistic H is asymptotically distributed as a chi-square with $k-1$ degrees of freedom.

4.3. Classical regression test

Finally, the equality of means may be tested considering the classical econometric test of lineal independence between the mean of largest Lyapunov exponents $\langle \lambda_{max}(T) \rangle$, in every sample size, and the sample size T . To that end, we have performed a linear regression

$$\langle \lambda_{max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T \quad \text{for } T = T_1, \dots, T_r = N,$$

obtaining the following statistical test:

$H_0 : \alpha_1 = 0$ *deterministic chaos*

$H_1 : \alpha_1 \neq 0$ *stochastic process*

5. Applications

In this paper we test for deterministic chaos on the simulated data used in the single-blind controlled competition among tests for nonlinearity and chaos provided by Barnett *et al.* (1997), applying the three versions of our test for stability of largest Lyapunov exponent with different the sample sizes.

As Barnett *et al.* (1997), we compute our tests twice: for small samples of 380 observations and for large samples of 2000 observations. For the 380 observations case, the subsamples sizes used in our tests were:

$$T_1 = 100, T_2 = 125, T_3 = 150, T_4 = 175, T_5 = 200, T_6 = 225, T_7 = 250, T_8 = 275, \\ T_9 = 300, T_{10} = 325, T_{11} = 350 \text{ and } T_{12} = 380.$$

For the 2000 observations case, the subsamples sizes used in our tests were :

$$T_1 = 100, T_2 = 250, T_3 = 500, T_4 = 750, T_5 = 1000, T_6 = 1250, T_7 = 1500, T_8 = 1750, \\ \text{and } T_9 = 2000.$$

First, we compute the mean values of 100 largest Lyapunov exponents using a moving blocks bootstrap procedure. The Lyapunov exponents were calculated using the algorithm proposed in Rosenstein *et al.* (1993). In Figures 2 and 3, respectively, we report the behaviour of mean values of Lyapunov exponents in front of several subsample sizes, and the distributions of 100 largest Lyapunov exponents for different sample sizes in the Barnett *et al.* (1997)'s series of 380 and 2000 observations.

As can be seen, only for the Feigenbaum series the mean $\langle \lambda_{max}(T) \rangle$ stabilises when the sample size T grows. Furthermore Figures 2 and 3 suggest that all distributions of largest Lyapunov exponents behave like a normal. Nevertheless, in Tables 1a,b,c,d,e and Tables 2a,b,c,d,e, we report the calculated values of Jarque-Bera test (distributed as a χ^2_2) obtaining from the empirical distributions of the largest Lyapunov exponent. The results suggest that normality is not rejected all cases, except for Feigenbaum series with parameter $c=3.57$. In subsection 5.1 we will show that the distribution of largest Lyapunov exponents is normal in Feigenbaum series with parameter $c=4$.

Tables 3a,b,c,d,e and Tables 4a,b,c,d provide the results of the Welch parametric t-tests for stability of largest Lyapunov exponent based on different the sample sizes.

As can be seen, the null hypothesis of mean equity (deterministic chaos) is rejected for all couple of samples, except for both Feigenbaum series of sample size 380 and 2000.

In Table 5, we report the results from the Kruskal-Wallis tests for stability of largest Lyapunov exponent with the sample size.

As shown in this table, the null hypothesis of deterministic chaos is again rejected in all cases, except for both Feigenbaum series.

Finally, in Table 6 we report the results for the classical regression version of our test for stability of largest Lyapunov exponent with the sample size.

According with this test, only model I (the Feigenbaum series) appears chaotic, being this conclusion reached with both the large and the small sample.

Therefore, all three versions of our test for stability of largest Lyapunov exponent with the sample size reject chaos for the GARCH, NLMA, ARCH and ARMA stochastic processes. In contrast, they do not reject chaos in the case of the Feigenbaum series. This is very favourable for our tests, since the latter is the only case of chaotic data.

Nevertheless, the case of the Feigenbaum series requires a supplementary analysis, which is done in the next subsection.

5.1. The Feigenbaum series

The Feigenbaum series proposed in Barnett *et al.* (1997) [that is $y_t = c y_{t-1}(1 - y_{t-1})$, $c = 3.57$, $y_0 = 0.7$] is really special as we will show. The problem is that the parameter $c = 3.57$ of this map is too closed to $c_\infty = 3.569946\dots$, the value of the parameter where the period 2^n ($n \rightarrow \infty$) cycle first occurs. For $c < c_\infty$ we have 2^n cycles and for $c_\infty \leq c \leq 4$ the map displays a rich variety of behaviours [see Jackson, 1989]. For $c > c_\infty$, except for the narrow bands where the solutions would oscillate on an n -cycle again (e.g. $n = 3$ for $3.83 < c < 3.86$), there are an infinite number of possible values for y_t that never repeats itself. For $c = 4$ fully developed chaos over the whole range from zero to one first appears in the Feigenbaum map.

For $c = 3.57 \cong c_\infty \cong 3.569946$ used by Barnett *et al.* (1997), the sequence generated by the Feigenbaum map is much less "regular" than a sequence with a

finite period of repetition. Nevertheless, the c_∞ sequence has an important difference with true chaotic behaviour because the c_∞ sequence is still marginally predictable in the sense that if two initial values are close enough to each other, the two sequences generated by Feigenbaum map, for these two initial conditions, will be very closed to each other even after a very long time. The reason is that at $c = c_\infty$, the infinite cycle is stable.

Finally, if we repeat calculations for $c = 4$, where the Feigenbaum map is fully chaotic, the largest Lyapunov exponents are closed to the theoretical value $\ln(2) \cong 0.6931$, with independence of initial values, as we report in Tables 7a and 7b.

6. Concluding remarks

Empirical research on detection of chaotic behaviour has expanded rapidly, but the results have tended to be inconclusive, due to lack of appropriate testing methods.

The general practice in the literature has been to take the existence of a positive Lyapunov exponent as an indication that the system is chaotic. Nevertheless, this condition is not sufficient for the detection of chaos, and does not help us to distinguish a chaotic process from a stochastic process. Indeed, any standard algorithm for calculating the largest Lyapunov exponent will find a finite, positive value for this exponent, both for chaotic as well as for stochastic processes.

In this paper, we combine the bootstrap statistical framework for hypothesis testing using the calculated Lyapunov exponents (proposed by Gençay, 1996), with the ergodic theory of deterministic dynamical systems in order to develop a new test to detect chaotic dynamics in time series. The new test is based on the stability of the mean in the distributions of the largest Lyapunov exponent calculated from different sample sizes, that is assured by Oseledec's (1968) theorem. This theorem provides a strong characteristic of chaotic deterministic processes that is not shared by stochastic processes. We show that while for (linear and nonlinear) stochastic processes the largest Lyapunov exponent increases with the sample size, for chaotic series the largest Lyapunov exponent is invariant when increasing the sample size.

We have applied this new test to the simulated data used in the single-blind controlled competition among tests for nonlinearity and chaos generated by Barnett *et al.* (1997), both for small samples (380 observations) and for large samples (2000 observations). The results suggest that the new test has high discriminatory power against interesting stochastic alternatives, both linear and nonlinear (GARCH, NLMA, ARCH and ARMA). Therefore, our test improves several tests available in the literature, since it has the ability to isolate the nature of the nonlinearity, stochastic or deterministic. In addition, our test behaves well in small samples.

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Figure 1.a: Largest Lyapunov exponents for different sample sizes.

Figure 1.b: Largest Lyapunov exponents for different sample sizes.

Figure 2: Distributions of the largest Lyapunov exponents for different sample sizes

Figure 3: Distribution of the largest Lyapunov exponents for different sample sizes.

Normality of largest Lyapunov exponents**Sample size=380**

Table 1a: Feigenbaum series (1)			
Sample size	Skewness	Kurtosis	Jarque-Bera test (2)
100	-0.35963	2.60319	2.75538
125	-0.99757	4.48991	25.31827
150	-1.29280	5.04117	44.31103
175	-1.52304	5.87980	71.75164
200	-1.27660	5.53923	52.94643
225	-0.61653	4.07475	10.92515
250	-0.84866	3.22227	11.96549
275	-0.91489	4.52203	23.13057
300	-0.65966	3.21444	7.29524
325	-0.80475	3.84513	13.49427
350	-1.42277	5.98701	69.49538
380	-1.06868	3.85391	21.63126

Notes:

(1) $y_t = 3.57 \cdot y_{t-1} \cdot (1 - y_{t-1})$ $y_0 = 0.7$.(2) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Table 1b: GARCH(1,1)

Sample size	Skweness	Kurtosis	Jarque-Bera test(1)
100	-0.41976	3.63361	4.51721
125	0.11672	2.95750	0.22991
150	0.28304	3.23393	1.53192
175	-0.43684	2.63820	3.65132
200	-0.11422	2.61127	0.83011
225	0.29881	2.13031	4.54688
250	0.05149	2.24308	2.38277
275	0.04521	2.38733	1.56614
300	0.10066	3.32029	0.58439
325	-0.07015	2.66147	0.54833
350	-0.15536	2.38712	1.92800
380	-0.02024	2.91759	0.03442

Note:

(1) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Table 1c: NLMA

Sample size	Skewness	Kurtosis	Jarque-Bera test (1)
100	0.06687	2.66923	0.51979
125	0.06507	2.39382	1.56957
150	-0.11476	4.13351	5.46158
175	0.14024	3.87090	3.41832
200	0.20843	3.15758	0.81098
225	0.08955	3.04615	0.13967
250	-0.04781	2.14387	3.03026
275	-0.41756	3.61861	4.41043
300	0.09997	2.36716	1.79858
325	-0.34795	3.24269	2.21794
350	-0.28622	2.70183	1.70104
380	0.10021	3.08465	0.19328

Note:

(1) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Table1d: ARCH(1)

Sample size	Skewness	Kurtosis	Jarque-Bera test (1)
100	-0.10272	3.08564	0.20230
125	-0.48380	3.35388	4.33441
150	-0.20691	3.06387	0.71589
175	0.09287	3.09956	0.18135
200	-0.12517	2.67837	0.67830
225	-0.18027	2.76476	0.75673
250	0.10209	3.15630	0.26998
275	0.16835	3.81380	3.16714
300	-0.08098	2.97513	0.10964
325	-0.04483	2.41263	1.44159
350	0.16019	3.17321	0.54163
380	-0.03789	2.48991	1.08591

Note:

(1) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Table 1e: ARMA(2,1)

Sample size	Skewness	Kurtosis	Jarque-Bera test (1)
100	-0.14286	3.08162	0.36056
125	-0.33992	2.63763	2.42340
150	-0.19963	3.22259	0.85326
175	0.15272	2.91518	0.41032
200	0.16567	3.47246	1.35976
225	0.11180	2.97783	0.20615
250	0.23905	2.52329	1.86131
275	-0.16656	2.95760	0.46047
300	0.24156	2.67466	1.38528
325	-0.37832	2.83538	2.44844
350	-0.07700	2.78459	0.28630
380	-0.06107	2.63375	0.60867

Note:

(1) Critical values for $\chi^2_{2,\nu}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Normality of largest Lyapunov exponents**Sample size =2000**

Table 2a: Feigenbaum series (1)			
Sample size	Skewness	Kurtosis	Jarque-Bera test (2)
100	-0.84641	3.81481	14.41237
250	-1.05537	4.70573	30.07252
500	-1.02604	4.14090	22.50997
750	-1.22260	5.51849	50.31401
1000	-0.92529	4.00456	18.10465
1250	-0.69836	3.78344	10.47216
1500	-0.89518	3.55784	14.35941
1750	-1.26703	4.76230	38.90257
2000	-0.97773	3.77358	18.05745

Notes:

(1) $y_t = 3.57 \cdot y_{t-1} \cdot (1 - y_{t-1})$ $y_0 = 0.7$.(2) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Table 2b: GARCH(1,1)

Sample size	Skewness	Kurtosis	Jarque-Bera test (1)
100	0.09272	2.76051	0.37461
250	0.01349	2.67356	0.43811
500	-0.27074	2.79354	1.37129
750	-0.20242	2.42968	1.99744
1000	-0.04508	2.89819	0.07552
1250	-0.35956	3.21153	2.29437
1500	-0.03533	3.03008	0.02408
1750	-0.10695	3.61461	1.72929
2000	-0.29413	2.95105	1.42286

Note:

(1) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Table 2c: NLMA

Sample size	Skewness	Kurtosis	Jarque-Bera test (1)
100	0.23801	2.71917	1.24726
250	-0.05936	2.48975	1.12068
500	-0.35175	2.60089	2.67127
750	0.18226	2.49746	1.57382
1000	-0.16732	2.68411	0.86471
1250	0.19216	2.55093	1.42658
1500	-0.22965	3.22394	1.06616
1750	-0.11202	2.95963	0.21161
2000	0.09430	2.64615	0.65654

Note:

(1) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Table 2d: ARCH(1)

Sample size	Skewness	Kurtosis	Jarque-Bera test (1)
100	0.25445	3.41780	1.77028
250	-0.05270	2.79258	0.22104
500	0.27991	3.37455	1.85254
750	0.11055	2.88655	0.25218
1000	-0.21853	3.77399	3.22617
1250	-0.08436	3.55295	1.36473
1500	-0.25134	3.24539	1.27771
1750	-0.12914	2.50152	1.28705
2000	-0.13303	3.28950	0.63129

Note:

(1) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Table 2e: ARMA(2,1)

Simple size	Skewness	Kurtosis	Jarque-Bera test (1)
100	-0.04789	2.39029	1.55542
250	0.12774	3.07986	0.29257
500	-0.05457	3.00669	0.04882
750	0.00702	2.47514	1.12569
1000	0.19017	2.57486	1.32874
1250	-0.02587	2.45184	1.23790
1500	-0.18933	2.58371	1.29312
1750	0.14225	2.68904	0.72537
2000	0.47872	3.03396	3.74786

Note:

(1) Critical values for $\chi^2_{2,\nu}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Tests for chaotic dynamics**Sample size=380**

Tabla 3a: Test for equality in the mean distribution of largest Lyapunov exponents. Feignbaum series (1)

Sample sizes	100	125	150	175	200	225	250	275	300	325	350	380
100	1.24149	0.57925	-0.19794	0.58980	0.54230	-0.67777	0.84445	-0.39948	0.11033	0.17715	-0.70430	
125		-0.58924	-1.33411	-0.54970	-0.80082	-1.82985	-0.44919	-1.60417	-1.13469	-0.96013	-1.82702	
150			-0.71838	0.02464	-0.12247	-1.17314	0.18572	-0.93502	-0.47799	-0.36533	-1.18704	
175				0.72586	0.69776	-0.43048	0.96928	-0.16782	0.29796	0.34411	-0.46261	
200					-0.14664	-1.16792	0.15408	-0.93558	-0.49141	-0.38133	-1.18229	
225						-1.23187	0.35299	-0.96007	-0.42318	-0.29386	-1.24087	
250							1.48421	0.28982	0.78092	0.78005	-0.04510	
275								-1.23494	-0.73143	-0.58001	-1.48768	
300									0.50729	0.53268	-0.32769	
325										0.07800	-0.80441	
350											-0.80334	
380												

Notes:

(1) Critical values for $N(\theta, I)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Table 3b: Test for equality in the mean distribution of largest Lyapunov exponents. GARCH(1,1) (1)

Sample size	100	125	150	175	200	225	250	275	300	325	350	380
100	-8.38009	-11.75099	-16.40917	-20.65693	-23.20905	-25.47963	-29.15033	-31.89103	-33.05731	-34.95659	-34.93817	
125		-3.99026	-9.35195	-14.42423	-17.39744	-20.10777	-24.67287	-28.01601	-29.55296	-31.90659	-31.95026	
150			-5.22301	-10.29302	-13.10033	-15.63670	-19.88020	-23.39608	-24.77856	-27.15658	-27.12026	
175				-5.25534	-7.97428	-10.41961	-14.52141	-18.37591	-19.68258	-22.19962	-22.09360	
200					-2.49405	-4.70145	-8.35768	-12.45690	-13.54862	-16.10184	-15.88341	
225						-2.20667	-5.88711	-10.30109	-11.39746	-14.10956	-13.85535	
250							-3.70923	-8.46744	-9.57496	-12.46242	-12.17284	
275								-5.47453	-6.59315	-9.83786	-9.47456	
300									-0.75865	-3.86298	-3.35089	
325										-3.28150	-2.72882	
350												0.64149
380												

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Table 3c: Test for equality in the mean distribution of largest Lyapunov exponents. NLMA (1)

Sample size	100	125	150	175	200	225	250	275	300	325	350	380
100	3.72689	-1.46566	-7.30268	-13.31516	-15.38818	-16.70120	-17.84357	-19.30487	-21.72181	-22.97397	-24.26358	
125		-5.75268	-12.40741	-19.20742	-21.90855	-23.61336	-24.47374	-26.60252	-30.18601	-31.81719	-32.55892	
150			-6.42598	-13.08384	-15.45005	-16.96430	-18.14028	-19.89124	-22.86206	-24.34935	-25.55888	
175				-6.77351	-8.91619	-10.31913	-11.83031	-13.31705	-15.90423	-17.34831	-18.98939	
200					-1.83320	-3.06927	-4.95066	-6.07988	-8.11077	-9.45305	-11.60742	
225						-1.25699	-3.32599	-4.43637	-6.48573	-7.90309	-10.25897	
250							-2.19604	-3.27320	-5.29600	-6.74911	-9.25027	
275								-0.88054	-2.53539	-3.83213	-6.40909	
300									-1.70150	-3.09070	-5.88274	
325										-1.53464	-4.74728	
350											-3.43084	
380												

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Table 3d: Test for equality in the mean distribution of largest Lyapunov exponents. ARCH(1) (1)

Sample size	100	125	150	175	200	225	250	275	300	325	350	380
100		-2.14031	-5.63828	-11.46177	-15.23656	-16.74813	-18.49436	-19.78816	-22.73146	-23.45690	-24.84118	-26.22713
125			-4.22054	-11.31831	-15.95791	-18.04216	-20.13316	-21.95616	-25.99719	-26.83484	-28.85810	-30.93032
150				-6.90330	-11.49901	-13.33445	-15.49843	-17.14458	-20.96046	-21.84906	-23.72332	-25.64487
175					-4.75728	-6.31814	-8.70776	-10.19895	-13.94953	-14.95895	-16.77132	-18.64820
200						-1.28755	-3.79109	-5.09550	-8.63533	-9.71741	-11.38459	-13.11705
225							-2.69642	-4.05580	-7.84563	-9.01592	-10.82749	-12.73383
250								-1.20285	-4.73355	-5.91965	-7.56274	-9.28658
275									-3.66031	-4.92417	-6.63792	-8.45344
300										-1.40920	-3.10205	-4.92137
325											-1.59704	-3.30830
350												-1.72814
380												

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Table 3e: Test for equality in the mean distribution of largest Lyapunov exponents. ARMA(2,1) (1)

Sample size	100	125	150	175	200	225	250	275	300	325	350	380
100	-0.15398	-1.71291	-3.38892	-4.53808	-8.88369	-13.37110	-17.03850	-18.17464	-18.53270	-18.55492	-21.15075	
125		-1.80859	-3.78715	-5.24110	-10.64545	-16.09666	-20.63854	-22.10107	-22.86569	-23.09099	-26.02595	
150			-1.75764	-2.93189	-7.72180	-12.72935	-16.83490	-18.12027	-18.59288	-18.66182	-21.51994	
175				-1.10927	-6.11787	-11.46331	-15.84149	-17.21610	-17.73632	-17.82119	-20.87227	
200					-5.48931	-11.42175	-16.30613	-17.86359	-18.57827	-18.75995	-22.06380	
225						-6.49577	-11.76343	-13.42100	-14.03320	-14.13546	-17.91745	
250							-5.12836	-6.67803	-6.86349	-6.70292	-10.82739	
275								-1.50111	-1.36822	-1.00491	-5.48757	
300									0.23200	0.66262	-3.98096	
325										0.45631	-4.51488	
350											-5.15619	
380												

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Tests for chaotic dynamics**Sample size =2000**

Table 4a: Test for equality in the mean distribution of largest Lyapunov exponents. Feigenbaum series (1)

Sample size	100	250	500	750	1000	1250	1500	1750	2000
100		0.15031	0.13438	-2.04829	0.45502	-0.64634	0.17970	-1.25196	0.25858
250			0.00052	-2.35165	0.35276	-0.85019	0.05705	-1.47532	0.14026
500				-2.11432	0.32120	-0.75872	0.05141	-1.34034	0.12680
750					2.29364	1.41710	2.05411	0.67630	2.14756
1000						-1.03417	-0.25791	-1.56564	-0.18972
1250							0.76901	-0.65424	0.85328
1500								-1.32460	0.07125
1750									1.40730
2000									

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Table 4b: Test for equality in the mean distribution of largest Lyapunov exponents. GARCH(1,1) (1)

Sample size	100	250	500	750	1000	1250	1500	1750	2000
100		-25.06362	-39.64827	-45.45503	-50.90325	-57.16468	-58.97172	-62.30481	-66.54763
250			-21.56103	-31.07175	-40.41020	-51.76574	-54.47664	-61.14995	-68.81687
500				-9.38774	-19.83189	-32.62292	-36.16640	-43.71386	-52.95072
750					-11.37845	-25.51476	-29.64754	-38.18083	-48.84444
1000						-13.98494	-18.75903	-27.01508	-38.19353
1250							-5.86281	-13.90711	-26.19270
1500								-7.04147	-18.51443
1750									-12.67149
2000									

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Table 4c: Test for equality in the mean distribution of largest Lyapunov exponents. NLMA (1)

Sample size	100	250	500	750	1000	1250	1500	1750	2000
100		-18.45967	-31.63916	-41.43096	-49.26358	-53.46519	-56.75038	-62.22199	-68.54333
250			-16.15247	-29.36122	-40.44287	-46.47567	-51.28861	-59.37835	-69.12686
500				-14.55236	-27.30398	-34.50427	-40.28246	-50.14606	-62.35471
750					-13.16660	-20.94881	-27.15492	-37.92762	-51.38094
1000						-8.19070	-14.65373	-26.07619	-40.39503
1250							-6.38661	-17.78176	-31.93458
1500								-11.50350	-25.72487
1750									-14.00362
2000									

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Table 4d: Test for equality in the mean distribution of largest Lyapunov exponents. ARCH(1) (1)

Sample size	100	250	500	750	1000	1250	1500	1750	2000
100		-24.02163	-37.47779	-46.37540	-53.27882	-57.94657	-60.26146	-63.79069	-67.27543
250			-15.39058	-27.26119	-36.97122	-43.62994	-47.00443	-52.17831	-57.31837
500				-14.29611	-26.84799	-35.69906	-40.29910	-47.43191	-54.59669
750					-12.89653	-23.54008	-28.50180	-36.47780	-44.64070
1000						-12.93662	-18.26362	-27.22148	-36.61378
1250							-4.38598	-12.09699	-20.26694
1500								-7.90760	-16.37806
1750									-8.61744
2000									

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Table 4e: Test for equality in the mean distribution of largest Lyapunov exponents. ARMA(2,1) (1)

Simple size	100	250	500	750	1000	1250	1500	1750	2000
100		-16.10762	-26.58134	-35.19607	-41.94218	-45.64488	-49.20723	-52.03070	-54.52210
250			-15.59012	-29.62440	-41.56799	-48.91401	-55.18961	-60.41342	-64.93825
500				-13.05387	-25.01520	-32.23050	-39.04067	-44.60847	-49.49129
750					-13.51422	-21.91900	-30.41786	-37.39844	-43.55895
1000						-8.19904	-17.84150	-25.63033	-32.61884
1250							-11.03102	-19.88977	-27.94593
1500								-8.57482	-16.51728
1750									-8.07677
2000									

Note:

(1) Critical values for $N(0,1)$; 2.576 (1%); 1.960 (5%); 1.645 (10%).

Tests for chaotic dynamics**Sample size=380****Kruskal-Wallis's non-parametric test**

Table 5: Kruskal-Wallis's test for equality in the mean distribution of largest Lyapunov exponents

Sample size	T=380 (1)	T=2000 (2)
Feignbaum series	10.93477	14.48629
GARCH(1,1)	900.65463	849.11469
NLMA	780.11469	852.88125
ARCH(1)	805.99018	843.69437
ARMA(2,1)	801.42094	850.06370

Notes:

(1) Critical values for $\chi^2_{11,v}$: 24.72 (1%); 19.68 (5%); 17.27 (10%).(2) Critical values for $\chi^2_{9,v}$: 21.67 (1%); 16.92 (5%); 14.68 (10%).

Test for stability of the largest Lyapunov exponent as function of sample size

Table 6: Test for equality of the largest Lyapunov exponents as function of sample size.

Linear regression $y = \alpha_0 + \alpha_1 x$

Sample size	T=380 (1)		T=2000 (2)	
Series	α_0	α_1	α_0	α_1
Feigenbaum	0.225257 (47.260167)	0.00003 (1.694421)	0.243959 (42.67107)	0.000001 (0.233717)
GARCH(1,1)	0.488313 (20.132180)	0.000977 (10.206105)	0.672528 (13.223819)	0.000221 (5.169461)
NLMA	0.517429 (22.503074)	0.000794 (8.752889)	0.682489 (19.639607)	0.0002098 (7.160317)
ARCH(1)	0.470792 (19.349901)	0.000983 (10.234567)	0.664688 (12.779747)	0.000235 (5.389933)
ARMA(2,1)	0.466566 (30.10877)	0.000764 (12.485107)	0.631055 (17.372245)	0.000199 (6.524499)

Note:

(1) Critical values for $t_{10,v}$; 3.169 (1%); 2.228 (5%); 1.812 (10%).(2) Critical values for $t_{7,v}$; 3.499 (1%); 2.365 (5%); 1.895 (10%).

Normality of largest Lyapunov exponents**Sample size=380**

Table 7a: Feigenbaum series (1)			
Sample size	Skewness	Kurtosis	Jarque-Bera test (2)
100	-0.20521	2.42030	2.06007
125	-0.28828	3.23824	1.58919
150	0.27580	2.55435	2.05337
175	-0.14968	2.80551	0.52037
200	-0.06721	2.36935	1.69782
225	-0.11062	2.42121	1.56777
250	0.08664	3.04609	0.13128
275	0.39641	2.26159	4.79308
300	0.11271	2.66877	0.65549
325	0.53918	2.97596	4.75077
350	0.42200	2.61622	3.51016
380	0.17271	2.35725	2.17414

Notes:

(1) $y_t = 4 \cdot y_{t-1} \cdot (1 - y_{t-1})$ $y_0 = 0.7$.(2) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).

Normality of largest Lyapunov exponents**Sample size =2000**

Table 7b: Feigenbaum series (1)			
Sample size	Skewness	Kurtosis	Jarque-Bera test (2)
100	-0.30599	2.82004	1.66153
250	0.08090	3.58922	1.52456
500	-0.44712	3.37944	3.85323
750	0.01413	3.12248	0.06452
1000	-0.03476	2.67500	0.45103
1250	-0.07765	2.40608	1.53882
1500	0.10611	2.79105	0.36216
1750	-0.26349	2.65413	1.62244
2000	0.02188	2.36220	1.66889

Note:

(1) $y_t = 4 \cdot y_{t-1} \cdot (1 - y_{t-1})$ $y_0 = 0.7$.(2) Critical values for $\chi^2_{2,v}$: 9.21 (1%); 5.99 (5%); 4.61 (10%).