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The Reduction of Dimension in the Study of Economic Growth Models*

J. R. Ruiz-Tamarit[†] and M. Ventura-Marco[‡]

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Abstract

We examine the dimension reduction method and prove that it could be misleading if we try to get some insight into the dynamics of the original system from the dynamics of the transformed system alone. The reduced system seemingly may give rise to a continuum multiplicity of steady states when, actually, it does exist a unique and isolated steady state or even it does not exist a steady state at all. We show how the dynamics for the primary variables that is recovered from the solution to the reduced system may be refuted by solving the original one. In our opinion there is no alternative because nothing can be regarded as a close substitute for the study of the original system. Although this method has been extensively used in studying different versions of the Lucas-Uzawa two-sector model, we will focus on one-sector models for checking its validity in the simplest way.

JEL classification: C61, C62, O41.

Keywords: Reduction of Dimension, Endogenous and Exogenous Growth, Multiplicity of Steady States, Transitional Dynamics.

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1 Introduction

Until a few years ago, almost every problem in growth theory was studied in terms of the levels of the most traditional macroeconomic variables. Indeed, output per capita, per capita physical and human capital stocks as well as per capita consumption, together with their growth rates, are the variables that have been the focus of the convergence literature for which the model of reference was always a sort of Ramsey-Solow model. However, after the emergence and rise of a lot of endogenous growth models, things changed in such a way that nowadays the standard procedure for studying modified Hamiltonian dynamic systems (MHDS), almost automatically, implies the reduction of dimension and the study of some new variables defined as a ratio between the older ones. Given that, we would like to know what can be said about the dynamics of the levels once the dynamics of the reduced system and the ratios are completely identified.

According to Lucas (1988), the procedure for studying MHDS arising from optimal growth models has to deal with the search for a closed-form solution for the primary variables before any transformation of the system be attempted. So, by solving the differential equations we will get the equilibrium paths which may be used empirically to compare with observations. In this article we are concerned with dynamic systems composed of accumulation equations, Euler equations and boundary conditions coming from intertemporal optimization problems, and specified only in terms of endogenous and decision variables like states, co-states or controls. Therefore, we do not mind reductions of dimension based on exogenous variables like population or disembodied technological change whose dynamic behavior is completely predetermined. Returning to the Lucas proposition, we need to point out that unless the original MHDS possesses a linear structure and is of small dimension, it may be difficult if not impossible to reach a closed-form solution. Hence, even if we cannot find its analytical solution, some kind of alternative analysis still needs to be developed to get information relative to the dynamic behavior of the primary variables involved in that system. We have to study in some way the issues of existence, unicity, dynamic stability as well as structural stability of the steady states. This is fundamental for getting an accurate characterization of the original system and, if possible, we can linearize and solve it explicitly. The problem is that the original non-linear MHDS emerging from endogenous growth models, when they are written in its state and co-state variables, have not a well-defined and isolated steady

state. So, we will face a major problem unless we can solve explicitly such a non-linear dynamic system.

The dimension reduction strategy has been widely applied. People define new variables as ratios between the variables of the original system -states, co-states and controls- in such a way that they transform the original system into another system with a lower dimension. Then, they study this transformed system and look for steady states -existence, unicity or multiplicity, dynamic stability- which will be interpreted as balanced growth paths from the original system. In fact, the search procedure for a balanced growth path in non linear MHDS has been simplified to finding a steady state for the corresponding dimension-reduced systems but, as we will see below, the steady states of the reduced system do not necessarily correspond to balanced growth paths into the original one.

In the case of Lucas' (1988) two-sector model with an externality depending on the human capital accumulation he only studied the properties of a particular solution trajectory which corresponds to the system's balanced growth path, leaving out of his analytical inquiry the remaining off-steady-state behavior of the system. However, an important stream of literature came later to prove the initial conjectures of Lucas relative to the transitional dynamics of the system. On the one hand, we have the well-known article by Mulligan and Sala-i-Martin (1993) which analyzes short run dynamics by means of the time elimination method. Their procedure builds upon a fundamental transformation of the 4-dimension original system by defining what they named control-like and state-like variables. Then, they characterize a stationary point and its stable manifold in the 3-dimension system which derives from the substitution of such new variables. The new system may be solved by using the conventional numerical method called shooting. However, under the additional assumption that the above stationary point corresponds to the balanced growth path which solves the original control problem, they apply the time elimination method that allows for a further transformation of the boundary value problem into a 2-dimension initial value problem. This is done by manipulating the policy functions and eliminating time. The dynamic system so generated will be finally solved by using easier numerical methods. On the other hand, Benhabib and Perli (1994), after writing the original MHDS in terms of states and controls, reduce the dimension by defining the same state-like and control-like variables as Mulligan and Sala-i-Martin. Then, they study the existence, unicity and multiplicity of steady states in such a reduced system, which are automati-

cally identified as balanced growth paths in the context of the original system. They also determine fundamental parameter constraints which guarantee interior steady states and use transversality conditions only to check that these stationary trajectories satisfy the non-explosivity conditions. As a peculiarity of their paper, they embark upon a local dynamic stability study of the reduced dynamic system by evaluating its Jacobian matrix and characterizing the solution space by means of the signs of the different eigenvalues. The system shows a unique solution of the saddle path type but, in some cases, it allows for a multiplicity of solutions because of the two free initial conditions and a unique positive eigenvalue.

The two previous papers use the dimension reduction method as a basic tool in their calculations. Now that this method is generalized, it seems that we are only concerned with the dynamic evolution of those new instrumental variables specified, for example, as the ratio between two per capita capital stocks, or the ratio of per capita consumption over per capita physical capital stock. But it is not so. Although these variables may suffice to study some economic problems, we are interested in the evolution of the MHDS original variables which coincide with the levels of per capita consumption, per capita physical capital stock and per capita human capital stock, because they are fundamental to get a complete description of the economic system. Moreover, given that the dimension reduction just depicted involves an important loss of information, the complete knowledge of the dynamics associated to the reduced model is not sufficient to deduce the true dynamics of the original variables.

The article is organized as follows. In section 2 we develop our hypothesis in the context of a general non-linear first order differential equation which will play the role of the reduced model. There, we analyze its dynamic features and introduce a feasible generalization of order two from which it could be derived. Then, we ask for the dynamics of such non-transformed variables which play the role of the original MHDS variables, and describe the different patterns we could find. In sections 3 and 4 we reduce the dimension of two different one-sector models of optimal growth, the Neoclassical model and the AK model. We describe the dynamics of each one and show how they simplify to the same one-dimension dynamic system analyzed in section 2. In section 5 we construct an artificial example to show that even when some additional information is available and may be used to break the previous indetermination, we are not completely free of the risk of mistakes. Finally, in section 6 we provide some conclusions.

2 The general non-linear reduced model

Let us start with the following non-linear differential equation:

$$\dot{z}(t) = -a \cdot z(t) + z(t)^2 \quad (1)$$

where $z(t)$ is a scalar time-dependent variable and $a \geq 0$ represents a general parameter. We assume that $z(t)$ is a non-predetermined variable. So, the initial value $z(t_0) = z(0)$ may be freely chosen. Consequently, we need a terminal (transversality) condition which will take the form: $\lim_{t \rightarrow \infty} z(t) = a$.

Consider first the case where the parameter takes the value $a > 0$, while the study of the alternative and less important case $a = 0$ may be found in the appendix. In this case, we have $\frac{\partial \dot{z}(t)}{\partial z(t)} = -a + 2 \cdot z(t)$, which implies a negative slope for $z(t) < \frac{a}{2}$ and a positive slope for $z(t) > \frac{a}{2}$. Moreover, $\frac{\partial^2 \dot{z}(t)}{\partial z(t)^2} = 2 > 0$, which implies global strict convexity with a minimum at $z(t) = \frac{a}{2}$. If we look for the steady state, we find two hyperbolic fixed point equilibria: $z_1^* = 0$ and $z_2^* = a$. That is, the roots of equation $\dot{z}(t) = 0 = z(t) \cdot (-a + z(t))$. The first one is locally asymptotically stable because at that point $\frac{\partial \dot{z}(t)}{\partial z(t)}(z(t) = 0) = -a < 0$, whereas the second one is locally asymptotically unstable because at that point $\frac{\partial \dot{z}(t)}{\partial z(t)}(z(t) = a) = a > 0$.

On the other hand, the logistic differential equation (1) may be solved in closed form by means of a simple transformation of variable according to Bernoulli's method. Then, a particular solution trajectory for this differential equation is:

$$z(t) = \frac{a \cdot z(0)}{(a - z(0)) \cdot e^{a \cdot (t-t_0)} + z(0)} \quad (2)$$

This trajectory converges to $z_1^* = 0^-$ for any $z(0) < 0$, takes the value $z(t) = z_1^* = 0 \ \forall t$ if $z(0) = 0$, and converges to $z_1^* = 0^+$ for any $0 < z(0) < a$. Moreover, $z(t) = z_2^* = a \ \forall t$ if $z(0) = a$, and diverges towards infinity for any $z(0) > a$ in a span of time shorter than $\frac{1}{a} \ln(\frac{z(0)}{z(0)-a})$. In this case, the transversality condition $\lim_{t \rightarrow \infty} z(t) = a$ implies $z(0) = a$.

Now, consider the above variable $z(t)$ being the result from a transformation of two primary variables, $x(t)$ and $y(t)$. In particular, assume that:

$$z(t) = \frac{y(t)}{x(t)} \quad (3)$$

Figure 1: Phase space and temporal trajectories.

with $x(t) > 0 \forall t$. Then, in spite of having a complete knowledge of the dynamic behavior of $z(t)$, we may be interested in the explicit dynamics of both $x(t)$ and $y(t)$ separately. In the context of the previous results characterizing the dynamics of the reduced model, the main point is to know what can be said about the dynamic characterization of the original variables in the x - y space.

We find different cases depending on the structural parameterization of the reduced dynamic system but, for the sake of simplicity and based on economic reasons, we will focus our attention on the case where $a > 0$, and then on the steady state $z^* = a$. This stationary point from the transformed system corresponds to the set of points which satisfy $y(t) = a \cdot x(t)$, $\forall t$ and $\forall x(t) > 0$, in the original system. However, this relationship may be satisfied for $x(t)$ and $y(t)$ following several *time-exponential* patterns of behavior:

- i) $x(t) = x^* = x(0)$ and $y(t) = y^* = y(0) \forall t$, being $y^* = a \cdot x^*$ and $y(0) = a \cdot x(0)$ because of the transversality condition.
- ii) $x(t) = x(0) \cdot \exp\{\lambda \cdot t\}$ and $y(t) = y(0) \cdot \exp\{\lambda \cdot t\}$ with λ , the common rate of growth, being positive or negative but constant in any case, and $y(0) = a \cdot x(0)$ because of the transversality condition.
- iii) $x(t) = x(0) \cdot \exp\{\int_0^t \lambda(s) \cdot ds\}$ and $y(t) = y(0) \cdot \exp\{\int_0^t \lambda(s) \cdot ds\}$ with $\lambda(t)$, the common rate of growth, being positive or negative, increasing or decreasing, and $y(0) = a \cdot x(0)$ because of the transversality condition.

In figure 2 we can see three different graphs which correspond to each one of the above patterns. The first one suggest that a continuum of globally unstable stationary equilibria does exist. In the second one, for a positive λ , we find that the variables $x(t)$ and $y(t)$ form a balanced growth path in a globally unstable environment. It is obvious that here the steady state position from the transformed system may obscure the existence of continuous growth with no transition. Finally, in the third one, for a positive but decreasing $\lambda(t)$, we may conclude that the variables $x(t)$ and $y(t)$ follow a balanced convergent path evolving towards a partially unstable stationary point, which is only reached when the variable growth rate $\lambda(t)$ becomes zero. In this case, the steady state position associated with the transformed system hides a monotonous transitory dynamics which corresponds to the original one. Therefore, the major problem we face is that we cannot decide between these alternative interpretations unless we have some additional information concerning the dynamic features of the system written in the two

Figure 2: Three alternative dynamic characterizations.

primary variables $x(t)$ and $y(t)$. For example, in terms of a general version of the Lucas-Uzawa growth model, Mulligan and Sala-i-Martin (1993) give a condition which is necessary, although not sufficient, for the existence of endogenous growth according to the second pattern of behavior previously described. However, keeping to their developments in that paper, we cannot exclude any of the other patterns we have just pointed out.

On the other hand, the consequences from a mistake in deciding between these patterns may be of first order. We only need to think about the consequences which would arise from considering that our variables are stationary, while in actual fact they are growing, at a constant or decreasing rate. For the same reason, we can imagine the magnitude of the error when we interpret that the variables are growing at a constant rate without transition, while true dynamics says that such variables are moving transitorily along a convergent path with a common decreasing growth rate. According to some models, policies that encourage and promote capital accumulation, particularly fiscal policies, produce only a transitional effect on growth. Therefore, improvements in factor efficiency that produce transitional effects will have permanent effects only on the long-run levels. On the other hand, according to other models, policies that enhance global factor efficiency also increase productivity growth, producing permanent effects on the levels as well as on the growth rates. These policies will contribute to accelerate long term growth.

In the next two sections, we are going to study two well known models of growth, each one with a very different (original) dynamics but practically equal reduced forms that will help us to illustrate the problem pointed out in the previous paragraphs.

3 The neoclassical optimal growth model

Consider the standard Ramsey–Cass-Koopmans model with exogenous technological progress. This model may be summarized by the following dynamic system defined in the state-control space:

$$\begin{aligned}\dot{\tilde{k}}(t) &= f(\tilde{k}(t)) - (\delta + n + x) \cdot \tilde{k}(t) - \tilde{c}(t) \\ \dot{\tilde{c}}(t) &= \sigma(t) \cdot [f'(\tilde{k}(t)) - \delta - \rho] \cdot \tilde{c}(t) - x \cdot \tilde{c}(t)\end{aligned}\tag{4}$$

with the initial condition: $\tilde{k}(t_0) = \tilde{k}(0)$ given, and the transversality condition: $\lim_{t \rightarrow \infty} e^{-(\rho - n - (1 - \frac{1}{\sigma(t)}) \cdot x) \cdot t} \cdot u'(\tilde{c}(t)) \cdot \tilde{k}(t) = 0$.

In these equations, $\tilde{k}(t)$ represents capital per unit of effective labor (predetermined) and $\tilde{c}(t)$ represents consumption per unit of effective labor (non-predetermined); $f(\cdot)$ denotes the production function written in intensive form which satisfies the Inada conditions; δ refers to the constant rate of capital depreciation; n and x are, respectively, the constant rates of population growth and labor-augmenting technological progress; $\sigma(t)$ represents the instantaneous intertemporal elasticity of substitution for consumption in the utility function; and ρ is the rate of discount for future utility, which is assumed to be greater than the population growth rate.

Under the usual assumptions of a CRRA instantaneous utility function which implies a constant intertemporal elasticity of substitution $\sigma(t) \equiv \frac{u'(c(t))}{-u''(c(t)) \cdot c(t)} = \sigma = \frac{1}{\Phi}$, and a technology represented by a Cobb-Douglas production function which implies a constant share for capital $\alpha = \frac{f'(\tilde{k}(t)) \cdot \tilde{k}(t)}{f(\tilde{k}(t))}$, the previous dynamic system may be simplified to:

$$\dot{\tilde{k}}(t) = \tilde{k}(t)^\alpha - (\delta + n + x) \cdot \tilde{k}(t) - \tilde{c}(t) \quad (5)$$

$$\dot{\tilde{c}}(t) = \frac{1}{\Phi} \cdot [\alpha \cdot \tilde{k}(t)^{\alpha-1} - \delta - \rho - \Phi \cdot x] \cdot \tilde{c}(t)$$

with the initial condition: $\tilde{k}(t_0) = \tilde{k}(0)$ given, and the transversality condition: $\lim_{t \rightarrow \infty} e^{-(\rho - n - (1 - \Phi) \cdot x) \cdot t} \cdot \tilde{c}(t)^{-\Phi} \cdot \tilde{k}(t) = 0$.

For the sake of simplicity, we will consider the particular case where $\Phi = \alpha$. According to Barro and Sala-i-Martin (1995), this dynamic system has a unique non-trivial steady state at $(\tilde{k}^*, \tilde{c}^*)$ with the property of saddle-path instability. The stable arm of this saddle-point structure corresponds to a ray passing across the origin and the steady state. When $\tilde{k}(0) < \tilde{k}^*$, along this convergent path, transitional dynamics implies that the positive growth rates $\gamma_{\tilde{k}}(t)$ and $\gamma_{\tilde{c}}(t)$, as well as $\gamma_k(t)$ and $\gamma_c(t)$, decline monotonically as the economy approaches the steady state. At the steady state we find that $\gamma_{\tilde{k}}^*$ and $\gamma_{\tilde{c}}^*$ are both constant and equal to zero, but γ_k^* and γ_c^* are both constant and equal to x . Thus, the variables per unit of effective labor $\tilde{k}^*$ and $\tilde{c}^*$ remain constant, while the per capita variables k^* and c^* keep growing at the constant rate of exogenous technological progress. Because of the assumption

Figure 3: Neoclassical optimal growth model.

$\Phi = \alpha$, during the transition the saving rate declines monotonically and we observe that $\gamma_{\tilde{k}}(t) = \gamma_k(t) - x = \gamma_c(t) - x = \gamma_{\tilde{c}}(t)$. The previous statements imply that the following relationship always holds.

$$c(t) = \left(\frac{\delta + \rho}{\Phi} - \delta - n \right) \cdot k(t) \quad (6)$$

This represents the equation of the unique stable (convergent) path, with $\frac{\delta + \rho}{\Phi} > \delta + n$ because of the transversality condition.

Despite the singularity of the case, as a consequence of the restrictive assumption regarding preferences, it is sufficient for clarifying the main hypothesis in this article. So, we proceed under such a useful, although unrealistic, assumption which allows us to perform an exact reduction of dimension. Define a new variable $z(t) \equiv \frac{\tilde{c}(t)}{\tilde{k}(t)} = \frac{c(t)}{k(t)}$, which by composition is non-predetermined, and substitute it in the dynamic system (5). The resulting one-dimensional dynamic system may be written as in the following first order differential equation:

$$\dot{z}(t) = - \left(\frac{\delta + \rho}{\Phi} - \delta - n \right) \cdot z(t) + z(t)^2 \quad (7)$$

This non-linear equation, which describes the dynamics for the variable $z(t)$, shares the same structure as equation (1) and may be solved in closed form according to the procedure explained in section 2. The transversality condition for such equation applied to the present context, $\lim_{t \rightarrow \infty} z(t) = \frac{\delta + \rho}{\Phi} - \delta - n$, is absolutely compatible with the transversality condition from the original Ramsey-Cass-Koopmans model after having substituted for the assumed variable transformation, if $\gamma_c(t)$ declines monotonically to zero as the economy approaches the steady state. Finally, we may conclude that the behavior of the variables $\tilde{k}(t)$ and $\tilde{c}(t)$ constitutes a particular example of the third pattern developed in section 2, and must not be confused with either of the other two cases.

4 The AK optimal growth model

In this section we consider the canonical AK optimal growth model as studied in Rebelo (1991). The main difference with respect to the neoclassical model refers to technology. We assume the absence of diminishing returns to capital. In this way the production function does not satisfy the Inada conditions because $\frac{f(k(t))}{k(t)} = f'(k(t)) = A$, constant $\forall t$. Moreover, we do not assume exogenous technological progress. Then, the variables $k(t)$ and $c(t)$ represent, respectively, capital per capita (predetermined) and consumption per capita (non-predetermined). It is also assumed $A > \delta + \rho$. Consequently, given $\rho > n$, we find that $A > \delta + n > 0$.

Now, under the usual CRRA instantaneous utility function, the model may be summarized by the following linear dynamic system in the state-control space:

$$\dot{k}(t) = (A - \delta - n) \cdot k(t) - c(t) \quad (8)$$

$$\dot{c}(t) = \frac{1}{\Phi} \cdot [A - \delta - \rho] \cdot c(t)$$

with the initial condition: $k(t_0) = k(0)$ given, and the transversality condition: $\lim_{t \rightarrow \infty} e^{-(\rho - n) \cdot t} \cdot c(t)^{-\Phi} \cdot k(t) = 0$.

The linear system has a unique steady state at $(k^*, c^*) = (0, 0)$ because of the assumptions $\frac{1}{\Phi} \cdot (A - \delta - \rho) \neq 0$ and $(A - \delta - n) \neq 0$. Nevertheless, this point does not make sense as an economic equilibrium since $k(0) > 0$. In addition, the Jacobian matrix has two positive eigenvalues indicating that the system is globally unstable. Every trajectory is a divergent path and, consequently, the next thing we can do is to identify the set of non-explosive trajectories which satisfy the transversality condition, for a given initial condition.

Solving the linear system and imposing the boundary conditions we get the following particular solution:

$$k(t) = k(0) \cdot \exp \left\{ \frac{A - \delta - \rho}{\Phi} \cdot (t - t_0) \right\} \quad (9)$$

$$c(t) = \left(A - \delta - n - \frac{A - \delta - \rho}{\Phi} \right) \cdot k(0) \cdot \exp \left\{ \frac{A - \delta - \rho}{\Phi} \cdot (t - t_0) \right\}$$

These equations determine a unique balanced growth path in the state-control space, which corresponds to a ray passing across the origin with a positive slope. Given $k(0) > 0$, the model has no transitional dynamics. The saving rate is constant and a constant positive economic growth rate $\gamma(t) = \gamma_k(t) = \gamma_c(t) = \frac{1}{\Phi} \cdot (A - \delta - \rho) \forall t$ exists. Thus, the per capita variables $k(t)$ and $c(t)$ grow indefinitely at the constant rate $\gamma(t)$ even in the absence of exogenous technological progress. The previous statements imply that the following relationship always holds.

$$c(t) = \left(A - \delta - n - \frac{A - \delta - \rho}{\Phi} \right) \cdot k(t) \quad (10)$$

This is the equation corresponding to the unique non-explosive path, with $A - \delta - n > \frac{A - \delta - \rho}{\Phi}$ because of the transversality condition.

Now, define as in the previous section the variable $z(t) \equiv \frac{c(t)}{k(t)}$, which is non-predetermined, and substitute in the dynamic system (8). Then, the resulting one-dimensional dynamic system may be written as the following first order differential equation:

$$\dot{z}(t) = - \left(\frac{\Phi - 1}{\Phi} \cdot (A - \delta) + \frac{\rho}{\Phi} - n \right) \cdot z(t) + z(t)^2 \quad (11)$$

This non-linear equation, which describes the dynamics for the variable $z(t)$, shares the same structure as both equation (1) from section 2 and

Figure 4: AK optimal growth model.

equation (7) from section 3. Further, it may be solved in closed form according to the method described in section 2. In this case the coefficient $\frac{\Phi-1}{\Phi} \cdot (A - \delta) + \frac{\rho}{\Phi} - n$ is always positive for any non-negative value of the parameter Φ . The adaptation of the transversality condition for such equations gives in the present context $\lim_{t \rightarrow \infty} z(t) = \frac{\Phi-1}{\Phi} \cdot (A - \delta) + \frac{\rho}{\Phi} - n$, which is perfectly congruent with the transversality condition from the original AK model under the assumed variable transformation. Finally, we may conclude that the behavior of the variables $k(t)$ and $c(t)$ in this model constitutes a particular example of the second pattern developed before in section 2, and must not be confused with any other.

In conclusion, we could find several alternative dynamic characterizations leading to the same reduced dynamic system, which necessarily has to be characterized by only one dynamic structure. Hence, an apparent indeterminacy arises when we know the dynamics from the reduced system but we do not know any feature from the original one. Any choice among the multiple available alternatives under these conditions may be risky and easily misleading.

5 A final paradigmatic example

In the two previous sections we have shown how two different original dynamics may be simplified to the same reduced dynamics. In this section, on the other hand, we will see how it is possible to conjecture a particular dynamics for the primary variables, starting from the solution to the reduced system and using partial and incomplete information about the original one, which is in apparent contradiction with the true dynamics that arises directly from the solution to the original system.

Consider the following dynamic system defined on the general x - y space:

$$\dot{x}(t) = \delta - \eta \cdot x(t) - y(t) \quad (12)$$

$$\dot{y}(t) = -\left(\frac{\beta}{\Phi}\right) \cdot y(t)$$

with the initial condition: $x(t_0) = x(0)$ given, and the transversality condition: $\lim_{t \rightarrow \infty} e^{-(\beta-\eta) \cdot t} \cdot x(t) \cdot y(t)^{-\Phi} = 0$. The variable $y(t)$ is non-predetermined and its initial value may be freely chosen. Assume $\delta > 0$, $\eta > 0$, $\beta > 0$ and $\beta > \eta$, as well as $\frac{\beta}{\Phi} > \eta$ for any $\Phi \in]0, \infty[$.

An immediate variable transformation, making $\tilde{x}(t) \equiv x(t) - \frac{\delta}{\eta}$, allows us to transform the system (12) into the following one:

$$\dot{\tilde{x}}(t) = -\eta \cdot \tilde{x}(t) - y(t) \quad (13)$$

$$\dot{y}(t) = -\left(\frac{\beta}{\Phi}\right) \cdot y(t)$$

with the initial condition: $\tilde{x}(t_0) = x(0) - \frac{\delta}{\eta}$ given, and the transversality condition: $\lim_{t \rightarrow \infty} e^{-(\beta-\eta) \cdot t} \cdot \left(\tilde{x}(t) + \frac{\delta}{\eta}\right) \cdot y(t)^{-\Phi} = 0$. This model may be exactly solved because of its simple structure but, according to the rationale of this exercise, we are going to ignore that point and we will behave as in the case of many economic growth models which are not explicitly solvable in closed form. Instead, we apply the general strategy which consists in the reduction of dimension. We define the variable $z(t) \equiv \frac{y(t)}{\tilde{x}(t)}$ and substitute in the dynamic system (13). Then, the resulting one-dimensional dynamic system may be written as follows:

$$\dot{z}(t) = -\left(\frac{\beta}{\Phi} - \eta\right) \cdot z(t) + z(t)^2 \quad (14)$$

where the variable $z(t)$ is a non-predetermined variable because of its dependence on $y(t)$, and its initial value $z(t_0) = \frac{y(t_0)}{x(t_0)} = \frac{y(t_0)}{x(0) - \frac{\delta}{\eta}}$ may be freely chosen.

This model can also be solved in closed form because of its particular structure according to Bernoulli's equation, but once again we will ignore that possibility. So then, if we look for balanced growth as usually occurs in the study of endogenous growth models, we realize that any reduced model like (14) might yield balanced growth paths, actually steady states, associated to the non trivial solutions to $\dot{z}(t) = 0$. Here, we consider it to be very important to use the conditional because of the specific indetermination pointed out along the previous sections. In general, the search procedure for a balanced growth path in modified Hamiltonian systems comes down to finding a steady state for the corresponding dimension-reduced systems, but we know that the steady states of the reduced system do not necessarily correspond to balanced growth paths in the original one, even if we impose the transversality conditions. As we have seen, the solution to this inconclusiveness will come from the study of the original non-transformed dynamic system, which we will do later.

For the moment, what we may conclude is that our reduced system has a unique non-trivial steady state which could be associated with a fundamental balanced growth path:

$$z(t) = z^* = \frac{\beta}{\Phi} - \eta > 0 \quad (15)$$

Many authors study in the context of the reduced dynamic systems different issues like existence, unicity or multiplicity, and dynamic stability properties of the steady states. As Xie (1994) pointed out, there are four different ways to study transitional dynamics and other related features of a dynamic system. Among the four approaches, only those which try to get the explicit solution, numerical or analytical, need to deal with the transversality conditions. In this section we are concerned more with the analytical closed form solutions than with numerical methods. To be precise, we are involved with the implications for the particular solutions which arise from the use of the transversality conditions, as well as with the general characterization of

the dynamic system. Coming back to the example we have to indicate that, connected with the dimension of the system itself, there is a big difference between our reduced system and those which come from several two-sector endogenous growth models in terms of the implied dynamics. Namely, while our simple steady state is easily identified as asymptotically unstable, for higher dimension systems it is still necessary to elucidate what kind of dynamic behavior the reduced system shows around its steady states, as done in Martínez-García (2001). Nevertheless, we consider that this matter is not crucial for the methodological goal chosen as the main subject in this paper.

Instead of the dynamics of the reduced model we are interested in the dynamics of the original variables. In the context of our example, given (15), we can ask about the behavior of the two intermediate variables $\tilde{x}(t)$ and $y(t)$. At first, if we look at section 2, any of the three possibilities referred to there may be applied here. To resolve this indetermination, a closer inspection of the second equation from system (13) might well be considered sufficient to decide between them. In that equation, the variable $y(t)$ decreases continuously at a constant rate. Consequently, the behavior of the variables $\tilde{x}(t)$ and $y(t)$ in this model seems to be one particular example of what may be called a balanced negative growth path. That is,

$$\frac{\dot{\tilde{x}}(t)}{\tilde{x}(t)} = \frac{\dot{y}(t)}{y(t)} = -\left(\frac{\beta}{\Phi}\right) < 0 \quad (16)$$

By integrating, we can recover the trajectory solutions for the primary variables $x(t)$ and $y(t)$ which may be easily written as follows:

$$x(t) = \frac{\delta}{\eta} + \left(x(0) - \frac{\delta}{\eta}\right) \cdot \exp\left\{-\left(\frac{\beta}{\Phi}\right) \cdot (t - t_0)\right\} \quad (17)$$

$$y(t) = \left(\frac{\beta}{\Phi} - \eta\right) \cdot \left(x(0) - \frac{\delta}{\eta}\right) \cdot \exp\left\{-\left(\frac{\beta}{\Phi}\right) \cdot (t - t_0)\right\}$$

These two functions should determine the dynamics for the primary variables arising from the stationary solution to the reduced system. In fact, this procedure parallels a generalized practice taken from the study of two-sector endogenous growth models. Nevertheless, we are going to finish this section by proving that the previous conjecture is absolutely wrong. The dynamic

system (12) may be analytically solved but, for the sake of simplicity, we prefer simply to check the compatibility of (17) with the transversality condition associated to that system. In doing so, we get the following result:

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-(\beta-\eta) \cdot t} \cdot \left[\frac{\delta}{\eta} + \left(x(0) - \frac{\delta}{\eta} \right) \cdot e^{-\left(\frac{\beta}{\Phi}\right) \cdot (t-t_0)} \right] \cdot y(t_0)^{-\Phi} \cdot e^{-\beta \cdot (t-t_0)} = \\ = \lim_{t \rightarrow \infty} e^{-\beta \cdot t_0} \cdot \frac{\delta}{\eta} \cdot y(t_0)^{-\Phi} \cdot e^{\eta \cdot t} \neq 0 \end{aligned} \quad (18)$$

We find that the recovered dynamics is in sharp contradiction with the true dynamics arising directly from the original system. The conjectured original dynamics give us the expression for a balanced (negative) growth path, but it does not satisfy the transversality condition and, consequently, it cannot be considered as a solution.

6 Conclusions

In the context of one-sector growth models we have examined the dimension reduction method, usually applied to study the different versions of the Lucas-Uzawa two-sector model, and proved that it could be misleading if we try to get some insight into the dynamics of the original MHDS from the dynamics of the transformed system alone.

We have shown how two different economic models with two disparate dynamic characterizations, the standard neoclassical exogenous growth model and the canonical endogenous growth model of the AK type, may be reduced to the same one-dimension differential equation. Both share the same structure and only differ in the terms included in their respective constant coefficients. Even though, they also share the sign of the coefficients because of the transversality conditions. So, two different original dynamics may be simplified to the same reduced dynamic model. In such a case, it could be possible to establish an empirical discrimination based on the value of the constant coefficient but, in general, we may conclude that unless we know something else about the dynamics of the original non reduced system, we cannot recover the dynamics of the original variables from the behavior of the transformed ones. The complete knowledge of the dynamics associated to the reduced model is not sufficient to deduce the true dynamics corresponding to the original one.

The reduction of dimension method entails a loss of information which is at the origin of a major indetermination. The reduced system may be interpreted mistakenly as giving rise to a continuum of steady states when, actually, there is only unicity or even absence of a well defined steady state. Moreover, we could be assuming that variables are stationary while actually they are growing, at a constant or decreasing rate. What is more, we might be thinking the variables are growing at a constant rate without transition, while true dynamics says that these variables are moving transitorily along a convergent path with a common decreasing growth rate. The major problem we face is that we cannot decide between these alternatives unless we have some additional information concerning the dynamic features of the original MHDS. In fact, the true dynamics for the original variables could remain unknown to us even when we have completely determined the dynamics for the transformed ones.

We also studied an example where the dynamics for the primary variables conjectured from the solution to the reduced system is contravened by the dynamics arising directly from the solution to the original one. The repeated use of MHDS taken from one-sector models of economic growth to illustrate our main hypothesis finds a justification in the fact that in this way we can derive the true dynamic behavior for the original variables and then compare with the dynamics conjectured from the well known solution to the reduced model.

7 Appendix

Here we study the alternative case mentioned in section 2. When $a = 0$ the dynamic equation (1) reduces to:

$$\dot{z}(t) = z(t)^2 \quad (19)$$

In this case, we have $\frac{\partial \dot{z}(t)}{\partial z(t)} = 2 \cdot z(t)$, which implies a negative slope for $z(t) < 0$ and a positive slope for $z(t) > 0$. Moreover, $\frac{\partial^2 \dot{z}(t)}{\partial z(t)^2} = 2 > 0$, which implies global strict convexity with a minimum at $z(t) = 0$. If we look for the steady state, we find that there is only one fixed point equilibrium: $z^* = 0$. However, this is not an hyperbolic equilibrium because at that point $\frac{\partial \dot{z}(t)}{\partial z(t)}(z(t) = 0) = 0$. Then, according to Verhulst (1990), this system is

structurally unstable and, consequently, $a = 0$ becomes a bifurcation point in the parameter space.

The closed form solution for equation (19) is:

$$z(t) = \frac{z(0)}{1 - z(0) \cdot (t - t_0)} \quad (20)$$

This trajectory converges to $z^* = 0$ for any $z(0) < 0$, takes the value $z(t) = z^* = 0 \forall t$ if $z(0) = 0$, and diverges towards infinity for any $z(0) > 0$ in a span of time shorter than $\frac{1}{z(0)}$. In this case, the transversality condition $\lim_{t \rightarrow \infty} z(t) = 0$ implies that $z(0)$ must be non-positive.

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Figure 5: Phase space and temporal trajectories.

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