**A New Test for Chaotic Dynamics
Using Lyapunov Exponents
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ABSTRACT

We propose a new test to detect chaotic dynamics, based on the stability of the largest Lyapunov exponent from different sample sizes. This test is applied to the data used in the single-blind controlled competition tests for nonlinearity and chaos that were generated by Barnett et al. (1997), as well as to several chaotic series. The results suggest that the new test is particularly effective when compared to other stochastic alternatives (both linear and nonlinear). The test size is one for large samples, although for small sample sizes it diminishes below the nominal size for two out of the three chaotic processes considered, what is not a surprise given some well-known properties of such processes.

JEL classification numbers: C13, C14, C15, C22

KEY WORDS: Chaos, Nonlinear Dynamics, Lyapunov exponents, Bootstrapping

1. Introduction

In a dissipative dynamical system, the existence of a positive Lyapunov exponent is usually taken as an indication of the chaotic character of the system. Lyapunov exponents provide information on the intrinsic instability of the trajectories of the system, and are computed as the average rate of exponential convergence or divergence of nearby trajectories in the phase space.

In recent years, there has been a burgeoning literature on the calculation of Lyapunov exponents for an unknown dynamical system reconstructed from a single time series. The seminal paper of Wolf *et al.* (1985) provides an algorithm to compute Lyapunov exponents in empirical applications, but this is sensitive to both the number of observations and the degree of noise in the data. More recently, however, some authors have proposed new methods for estimating Lyapunov exponents that perform well even for small samples [see, among others, Dechert and Gençay (1992), Abarbanel *et al.* (1991, 1992), and Rosenstein *et al.* (1993)].

There are many papers that use Lyapunov exponents to detect chaotic dynamics in financial time series, especially in exchange rate series. Earlier examples of research in this area include Bajo-Rubio *et al.* (1992) and Dechert and Gençay (1992), where Lyapunov exponents are used to distinguish between linear, deterministic processes (with negative Lyapunov exponents) and nonlinear, chaotic deterministic processes (where the largest Lyapunov exponent is positive). These and other papers have been criticised for the absence of a distributional theory providing a statistical framework for hypothesis testing using the Lyapunov exponents that are calculated. In this sense, Gençay (1996) presents a methodology to compute the empirical distributions of Lyapunov exponents using a blockwise bootstrap technique. This methodology provides a formal test of the hypothesis that the largest Lyapunov exponent equals some hypothesised value, and can be used to test for chaotic dynamics. The test proposed by Gençay (1996) is particularly useful in those cases where the largest Lyapunov exponent is positive, but very close to zero. More recently, Bask and Gençay (1998) use the same statistical framework to provide a test for the presence of a positive Lyapunov exponent in an observed time series. The numerical examples show that both the Gençay (1996) and the Bask and Gençay (1998) test statistics behave well for small samples. These papers have been very influential from an empirical point of view and, for example, Bask (1998), using the test suggested by Bask and Gençay (1998), finds evidence that some exchange rates can be characterised by deterministic chaos.

Despite the growing interest in the econometric literature aimed at distinguishing between non-linear deterministic processes and non-linear stochastic processes, there remain important disagreements and controversy about the available results. A key paper in this area is Barnett *et al.* (1997), where some data series were simulated from different generating models in order to evaluate the behaviour, both for large and small samples, of five highly regarded tests for nonlinearity or chaos. The tests considered in that paper are the Hinich bispectral test (Hinich, 1982), the BDS test (Brock *et al.*, 1996), the NEGM test (Nychka *et al.*, 1992), the White test (White, 1989), and the Kaplan test (Kaplan, 1994). The results concerning the power function of some of these tests proved to be rather surprising, since none of them had the ability to isolate the origins of the nonlinearity or chaos to within the structure of the economy.

The aim of this paper is to propose a new test for the presence of chaos, based on the behaviour of the estimated Lyapunov exponents, for different sample sizes. As we shall try to illustrate, while the largest exponent of a chaotic process is more or less stable with respect to sample size (showing a stationary behaviour when the sample size increases), the largest Lyapunov exponent of a stochastic process is not. Therefore, we suggest testing chaotic dynamics by estimating the empirical distributions of the largest Lyapunov exponents for different sample sizes and comparing their means. The proposed new test proves to be very effective when compared to stochastic processes, hence providing further refinement over those of Gençay (1996) and Bask and Gençay (1998).

The rest of the paper is organised as follows. Section 2 presents the statistical framework used in the paper. Section 3 discusses the stability of the largest Lyapunov exponent with sample size. Section 4 proposes the new test for distinguishing chaos from random behaviour. Section 5 reports the results of applying our test to several chaotic processes, as well as to the simulated data used in the single-blind controlled competition tests performed by Barnett *et al.* (1997). Section 6 presents a comparison with Bask and Gençay's (1998) test. Finally, Section 7 provides some concluding remarks.

2. A statistical framework for testing chaotic dynamics via Lyapunov exponents

In order to examine the properties of a deterministic dynamical system we make use of ergodic theory, since it provides a statistical framework where different degrees of the complexity of attractors and motions can be distinguished [see Eckmann and Ruelle (1985) for a survey]. Furthermore, ergodic theory allows us to describe the time averages of a dynamical system and to consider when transients become irrelevant. Once transients are completed, the motion of the dynamical system typically settles near a subset of $\mathfrak{R}^n$, called an *attractor*. In the particular case of dissipative systems, where the phase-space volumes are concentrated by time evolution, the volume occupied by the attractor is in general very small in relation to the phase space. Even if a system contracts its volume, it does not mean that its length is contracted in all directions: some directions may be stretched and some directions contracted. This implies that, even in a dissipative system, the final motions may be unstable within the attractor. This instability usually manifests itself in *sensitive dependence on initial conditions*, which means an exponential separation of orbits (as time goes on ¿ELAPSES?) of points that were initially very close each other on the attractor. In this case, we say that the attractor is a *strange attractor* and that the system is *chaotic*.

In ergodic theory, statistical averages can be computed either in terms of time averages or space averages. Let us consider, for simplicity, a discrete dynamical system of dimension n , $\bar{x}_{t+1} = \bar{F}(\bar{x}_t)$, where $F : \mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ is a vectorial differentiable function. The *time average* of a function φ along a (forward) trajectory $\bar{x}_i$ with initial condition $\bar{x}_0$, of a discrete dynamical system is defined by

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N-1} \varphi(\bar{x}_i)$$

In a similar way for a continuous flow ϕ_t , arising from a continuous dynamical system $\frac{d\bar{x}}{dt} = \bar{F}(\bar{x})$, the time average of a function along a (forward) trajectory is

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(\phi_t(x)) dt$$

Time averages often depend on initial conditions. However, when the dynamical system has an attractor, all trajectories have the same statistical properties.

A measure of complexity in chaotic motion may be obtained by analyzing the sensitivity of the dynamical behaviour to the initial conditions given by two infinitely close initial states. For chaotic systems, points in a common neighbourhood in the phase space separate exponentially with time. Let us illustrate the basic idea by means of a discrete dynamical system of dimension n , $\bar{x}_{t+1} = \bar{F}(\bar{x}_t)$. In order to examine the stability of the trajectories of the system, let us consider how the system amplifies a small difference between the initial conditions $\bar{x}_0$ and $\bar{x}'_0$:

$$\bar{x}_T - \bar{x}'_T = \bar{F}^T(\bar{x}_0) - \bar{F}^T(\bar{x}'_0) \cong D\bar{F}^T(\bar{x}_0)(\bar{x}_0 - \bar{x}'_0)$$

where $\bar{F}^T(\bar{x}_0) = \bar{F}(\bar{F}(\dots\bar{F}(\bar{x}_0)\dots))$ denotes the T successive iterations of the dynamical system starting from the initial condition $\bar{x}_0$, and where $D\bar{F}^T(\bar{x}_0)$ is the Jacobian of the function $\bar{F}(\bar{x})$.

By the chain rule, we have

$$D\bar{F}^T(\bar{x}_0) = D\bar{F}(\bar{x}_{T-1})D\bar{F}(\bar{x}_{T-2})\dots\dots\dots D\bar{F}(\bar{x}_0)$$

In this context, the Lyapunov exponents are defined as follows (Guckenheimer and Holmes, 1990): Let us consider the family of subspaces $V_i^{(1)} \supset V_i^{(2)} \supset \dots \supset V_i^{(n)}$ in the tangent space at $\bar{F}^i(\bar{x})$ and the numbers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ with the properties that:

$$(1) D\bar{F}(V_i^{(j)}) = V_{i+1}^{(j)}$$

$$(2) \dim V_i^{(j)} = n + 1 - j$$

$$(3) \lim_{T \rightarrow \infty} \frac{1}{T} \ln ||\sqrt{(D\bar{F}^T)^* \cdot (D\bar{F}^T)}(\bar{x}_0)|| = \lambda_j \text{ for all } \bar{x}_0 \in V_0^{(j)} - V_0^{(j+1)}, \text{ where } (D\bar{F}^T)^*$$

is the transpose of $D\bar{F}^T$

Then, the real numbers λ_j are called the *Lyapunov exponents* of F at $\bar{x}_0$. Lyapunov exponents offer information on how orbits on the attractor move apart (or together) given the dynamic evolution of the system. One can also define them by the rate of stretching or shrinking of line segments, areas, and various dimensional subvolumes in the phase space. Line segments grow or shrink by a factor of $e^{t\lambda_1}$, areas by a factor of $e^{t(\lambda_1+\lambda_2)}$ and so forth. If one or more of the Lyapunov exponents are positive, then we have *chaos* in the motion

of the system. The sum of the Lyapunov exponents is negative ($\lambda_1 + \lambda_2 + \dots + \lambda_n < 0$) for dissipative systems [see Abarbanel (1996)].

The possibility of obtaining, in a deterministic dynamical system, Lyapunov exponents that are representative of short-run divergences in trajectories with many closed initial points is based on Oseledec's (1968) *multiplicative ergodic theorem*. If we assume that there exists an ergodic measure of the system, this theorem justifies the use of arbitrary phase space directions when calculating the largest Lyapunov exponent. In this case, the Lyapunov exponents have a mean in a global sense, allowing the complexity of a deterministic dynamical system of dimension n to be characterised simply by n real numbers.

Oseledec's (1968) multiplicative ergodic theorem states that, under quite general conditions on the function $\bar{F}$, the limit of expression (3) does exist for almost all $\bar{x}_0$ (with respect to the invariant measure μ) and is independent of the initial condition $\bar{x}_0$ considered (except for a set of null measure). Therefore, the multiplicative ergodic theorem implies that the Lyapunov exponents are invariant numbers representing "globally" the complexity of the dynamical system under study, independently of the initial condition considered. In this sense it is also important to point out that Gençay and Dechert (1992), Gençay and Dechert (1996) and Dechert and Gençay (2000), have studied the topological invariance of the Lyapunov exponent estimator from observer dynamics.

Oseledec's theorem is based on the ergodic theory of deterministic dynamical systems and justifies the use of arbitrary phase space directions when calculating the largest Lyapunov exponents. Nevertheless, as both Whang and Linton (1999) and Tong (1990) point out, Lyapunov exponents can be interpreted within the standard non-linear time series framework as a measure of local stability and is of interest even without any direct connection with deterministic chaos.

There are several suitable estimation methods in order to obtain Lyapunov exponents based on kernels, nearest neighbours, splines, local polynomials and neural nets [see Härdle and Linton (1994) for a general discussion]. McCaffrey *et al.* (1992) distinguish two classes of methods for estimating the largest Lyapunov exponent λ_{max} : (i) Direct methods like Wolf *et al.* (1985) or Rosenstein *et al.* (1993), which assume that the initial divergence $(\bar{x}_0 - \bar{x}'_0)$ grows at the exponential rate given by λ_{max} in the reconstructed state space of a time series; and (ii) Jacobian methods, where data are used to estimate the Jacobians from an estimation of the conditional expectation of the

process, which allows λ_{max} to be estimated. Examples of Jacobian methods are those proposed by MacCaffrey *et al.* (1992), Nychka *et al.* (1992) or Gençay (1996). Although both conceptions (direct and Jacobian methods) agree with respect to the conception of estimated Lyapunov exponents for chaotic nonlinear deterministic processes, their approach to the Lyapunov exponents of stochastic processes are extremely different.

On one hand, direct methods act directly on the time series to estimate λ_{max} and are not intended, *ex ante*, to separate the effect of the random variables into the series. Direct methods are based on the philosophy that chaos provides a link between determinism and randomness; in this view the dimension and λ_{max} of IID noise, in theory, is infinite, and if the deterministic definition of the Lyapunov exponents is taken literally, the Lyapunov exponents are also infinite in presence of noise. Schuster (1996, p. 112) or Eubank and Farmer (1990, p. 160) support this conception where stochastic processes will have, necessarily, positive λ_{max} because of the infinite dimensionality of the noise.

On the other hand, in order to estimate the Lyapunov exponents in stochastic processes, Jacobian methods use nonparametric regression tools (Kernels, neural networks, etc.) trying to isolate the deterministic conditional mean of the process and estimate the derivatives in order to reconstruct the Jacobian. In this conception stochastic processes may have λ_{max} negative, for instance stationary linear autoregressions have $\lambda_{max} < 0$, while for unit root process $\lambda_{max} = 0$ and only for explosive linear autoregressions $\lambda_{max} > 0$ (See Whang and Linton (1999), p 5) .

Therefore, within the theory of dynamical systems, a chaotic system is characterized by globally bounded trajectories in the phase space with a positive largest Lyapunov exponent, while in the frame of direct methods to estimate λ_{max} , a white noise process has (in theory) an infinite largest Lyapunov exponent [see Schuster (1996) or Eubank and Farmer (1990)].

Nevertheless, in practical implementations, using finite time series, any standard algorithm for calculating the largest Lyapunov exponent, with direct methods, will find a finite positive value for this exponent, for any random process. Therefore, the largest Lyapunov exponent on its own is not able to distinguish between a chaotic, non-linear deterministic process and a random process. This problem is especially relevant in financial time series, where non-linear stochastic processes, such as GARCH processes, are usually postulated as alternative models to the chaotic behaviour [see, e. g., Hsieh (1991)].

Gençay (1996) proposed a statistical framework for testing chaotic dynamics using a moving blocks bootstrap procedure.

Consider a sequence $\{X_1, X_2, \dots, X_N\}$ of weakly dependent stationary random variables, and let $\{x_1, x_2, \dots, x_N\}$ be a time series realisation of this stochastic process. According to Künsch (1989) and Liu and Singh (1992), the distribution of certain estimators of interest can be consistently constructed by applying a moving blockwise bootstrap. Let $B_t^d = \{x_t, x_{t-1}, \dots, x_{t+d-1}\}$ denote a moving block of d consecutive observations, where $t \leq N - d + 1$. For a time series of N elements, we can form a set $\{B_1^d, \dots, B_{N-d+1}^d\}$ of blocks with length d . Let $k = \text{int}(N/d)$ [where $\text{int}()$ denotes the integer part], so that by sampling with replacement of k blocks denoted by $\{B_{i_1}^d, \dots, B_{i_k}^d\}$ we will form the bootstrap sample.

In order to obtain the sample distribution of the largest Lyapunov exponent $\lambda_{\max}$, we will repeat this procedure to construct a sequence of sub-families of k blocks taken with replacement from the family of d -dimensional blocks $\{B_1^d, \dots, B_{N-d+1}^d\}$ that can be generated with the time series $\{x_1, x_2, \dots, x_N\}$. For each sub-family of k blocks, we can apply some standard procedure to compute the largest Lyapunov exponent $\tilde{\lambda}_{\max}$ by taking the pairs of nearest neighbours from each sub-family of blocks. Repeating this process a large number of times, we will obtain the empirical distribution of the largest Lyapunov exponent $\tilde{\lambda}_{\max}$.

At this point it is important to observe the difficulty of Jacobian methods for estimating $\lambda_{\max}$ within the framework of bootstrap replications. As Ziehmann *et al.* (1999) pointed out, a bootstrap algorithm must be used with caution if Lyapunov exponent estimates rely on the product of matrices because matrix multiplication does not commute, except in one dimension. In order to avoid such complications with the product of Jacobians along the trajectory, we use a simple direct method for estimating the largest Lyapunov exponent $\lambda_{\max}$ of a time series proposed by Rosenstein *et al.* (1993). Given that the divergence between the nearest neighbours takes place at a rate approximated by the largest Lyapunov exponent, Rosenstein *et al.* suggest the choice of a pair of neighbours as initial conditions for different trajectories, and to estimate $\lambda_{\max}$ by averaging the exponential divergences of these initially close state-space trajectories.

Rosenstein's method may be outlined as follows. Let us consider an observed time series $\{x_1, x_2, \dots, x_N\}$. Following Takens (1981) theorem, we start by reconstructing the phase-space vector using the blocks $\{B_1^d, \dots, B_{N-d+1}^d\}$ defined above, where $B_t^d = \{x_t, x_{t-1}, \dots, x_{t+d-1}\}$. For each point B_t^d , we search for the nearest

neighbour point B_t^d in the reconstructed phase space that minimises the distance to that reference point:

$$d_t(0) = \min_{B_t^d} \|B_t^d - B_t^d\| ,$$

where $\|\dots\|$ denotes the Euclidian norm. To consider each pair of neighbours as initial conditions for different trajectories, the temporal separation between them should be greater than the mean period of the time series:

$$|t - t^*| > \text{mean period}$$

This mean period can be estimated as the reciprocal of the mean frequency of the power spectrum of the time series under study.

The divergence between the nearest neighbours B_t^d and B_t^d takes place at a rate approximated by the largest Lyapunov exponent:

$$d_t(i) \cong d_t(0) \exp(\lambda_{\max} i \Delta t) \text{ for } i = 1, \dots, T \quad (1)$$

where i is the number of discrete-time steps following the nearest neighbour, Δt is the sampling period of the time series, and $d_j(i)$ is the distance between the j th pair of nearest neighbours after i discrete time steps. (Recall that $i\Delta t$ corresponds to seconds so that Lyapunov exponents are expressed in bits/second). Taking the logarithm of both sides of this last expression, we obtain:

$$\log(d_t(i)) \cong \log(d_t(0)) + \lambda_{\max} i \Delta t \quad (2)$$

For each value of t between 1 and $N-d+1$, this equation represents a set of approximately parallel lines, each with a slope that is approximately proportional to $\lambda_{\max}$. The largest Lyapunov exponent is then estimated using a least-square fit with a constant to the average line defined by $\langle \log(d_t(i)) \rangle$, where $\langle \rangle$ denotes the average value over all values of t .

There exist several key parameters in the Rosenstein *et al* algorithm. Besides the embedding dimension (d) (that will be called the moving-block length for the moving blocks bootstrap procedure), we have to select the lag or reconstruction delay, the mean period and the number of discrete-time steps (i) allowed for divergence between nearest neighbours B_t^d and B_t^d in the phase space. We explain later how to select these parameters following the author's recommendations.

3. Stability of largest Lyapunov exponents with the sample size for chaotic processes

From a theoretical point of view, the reason for the stability of the largest Lyapunov exponent with respect to the sample size can be found in Oseledec's (1968) theorem, which states that for a large enough sample size, these exponents will converge to some stable values associated with the complexity of the attractor.

For chaotic time series, Oseledec's theorem guarantees the possibility of making short-run forecasts based on the reconstructed phase space. The Lyapunov exponents are nothing but a measure (in exponential scale) of the mean forecast errors using the nearest neighbour points in the phase space. However, when analysing a time series generated by a non-deterministic stochastic process, nothing guarantees the stability of the Lyapunov exponents. Oseledec's theorem only affects deterministic processes via ergodic theory. For a stochastic process, as the number of observations increases, the variability of the largest Lyapunov exponent will increase and, therefore, the largest Lyapunov exponent itself will also increase without limit with the sample size.

As we shall see, our simulations show an essential difference between chaotic and stochastic processes via Lyapunov exponents. If we want to reconstruct trajectories of a time series in a phase space that are sampled from a stochastic process, there is no guarantee of convergence in any algorithm towards the largest Lyapunov exponent, because the Lyapunov exponents are not necessarily stable and independent of the initial conditions and sample size. For stochastic processes, the algorithm is only able to estimate *local Lyapunov exponents*. Local Lyapunov exponents are a measure of the local stability of the process and may be highly dependent on the sample size and the initial condition considered.

Our simulations are based on different stochastic and chaotic processes. First of all, and following Barnett et al. (1997), let us consider samples of size 380 and 2000 observations of the following five models:

(i) A fully deterministic, chaotic Feigenbaum recursion of the form:

$$y_t = 3.57y_{t-1}(1 - y_{t-1}),$$

where the initial condition was set at $y_0 = 0.7$ ¹.

(ii) A GARCH process of the following form:

$$y_t = h_t^{1/2} u_t,$$

where h_t is defined by

$$h_t = 1 + 0.1y_{t-1}^2 + 0.8h_{t-1},$$

with $h_0 = 1$ and $y_0 = 0$.

(iii) A nonlinear moving average (NLMA) process:

$$y_t = u_t + 0.8u_{t-1}u_{t-2}.$$

(iv) An ARCH process of the following form:

$$y_t = (1 + 0.5y_{t-1}^2)^{1/2} u_t,$$

with the value of the initial observations is set at $y_0 = 0$, and

(v) An ARMA model of the form:

$$y_t = 0.8y_{t-1} + 0.15y_{t-2} + u_t + 0.3u_{t-1}, \text{ with } y_0 = 1 \text{ and } y_1 = 0.7.$$

With the four stochastic models, the white noise disturbances u_t , are sampled independently from a standard normal distribution. Note that of the five generating models, only model (i) is chaotic.

In order to provide further, and stronger, evidence supporting our claim that the observed invariance property of the largest Lyapunov exponent holds for all chaotic processes, we also consider the Hénon map and the Lorenz attractor.

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¹ The Feigenbaum series proposed in Barnett *et al.* (1997) [i. e., $y_t = c y_{t-1}(1 - y_{t-1})$, $c = 3.57$, $y_0 = 0.7$] is quite special. The problem is that the parameter $c = 3.57$ of this map is too close to $c_\infty = 3.569946\dots$, the value of the parameter where the period 2^n ($n \rightarrow \infty$) cycle first occurs [see Jackson, 1989]. For $c < c_\infty$ we have 2^n cycles and for $c_\infty \leq c \leq 4$ the map displays a rich variety of behaviours [see Jackson, 1989]. For $c > c_\infty$, except for the narrow bands where the solutions would oscillate again according to an n -cycle (e.g. $n = 3$ for $3.83 < c < 3.86$), there is an infinite number of possible values for y_t that never repeat themselves.

For $c = 3.57 \cong c_\infty \cong 3.569946$ (used by Barnett *et al.* (1997)), the sequence generated by the Feigenbaum map is much less "regular" than a sequence with a finite period of repetition. Nevertheless, the c_∞ sequence has an important difference with true chaotic behaviour. The reason is that the c_∞ sequence is still marginally predictable in the sense that if two initial values are close enough to each other, the two sequences generated by Feigenbaum map, for these two initial conditions, will be very closed to each other even after a very long time. This is so because, at $c = c_\infty$, the infinite cycle is stable.

(vi) The Hénon (1976) map is described by the following system:

$$\begin{aligned}x_{t+1} &= 1 - 1.4x_t^2 + y_t \\ y_{t+1} &= 0.3x_t\end{aligned}$$

with the initial points $x_0 = 0.5$ and $y_0 = 0.2$.

(vii) The well-known Lorenz (1963) attractor is the three-dimensional continuous-time system:

$$\begin{aligned}\dot{x} &= 10(y - x) \\ \dot{y} &= x(28 - z) - y \\ \dot{z} &= xy - \frac{8}{3}z\end{aligned}$$

Lorenz's system was solved using a straightforward fourth-order Runge-Kutta method, resulting a sampling period in the resolution of the system approximately of $\Delta t \cong 0.01$. Considering the average mutual information $I(T)$ for the signal $x(t)$ obtained after integration of the Lorenz's system, the minimum of this function is at $T = 10$; following Abarbanel (1996), a time lag $\tau = 10$ was used in order to obtain a series $x(t_0 + n\tau)$, $n = 1, \dots, 2000$ as is usual for the phase reconstruction. The initial point was chosen near the attractor and transient points were discarded.

We calculated the largest Lyapunov exponent applying the algorithm proposed by Rosenstein *et al.* (1993) to the time series generated by these models for each sample size between 200 and 2000 taking increments of 20 observations (i. e., 200, 220, 240, ..., 2000). Figure 1 displays the results of estimating the largest Lyapunov exponents λ_{max} for the simulated data series used in Barnett *et al.* (1997) and for the two new chaotic series (Hénon map and Lorenz attractor) for different sample sizes, from 200 to 2000 observations, for a moving block of size $d=3$. Other moving block sizes show similar results.

Figure 1

Given the evidence presented in Figure 1, the existence of a positive largest Lyapunov exponent does not imply the presence of chaos in a given time series. However, Figure 1 shows an interesting and essential difference between chaotic and stochastic processes. While the largest Lyapunov exponent in the deterministic models stabilises (in some cases even slightly decreases) as the sample size increases, for all the stochastic processes, the largest Lyapunov exponent always increases with the sample size. The stability of $\hat{\lambda}_{max}$ with the sampling size for chaotic processes, versus the positive relationship for stochastic processes, appears to be an essential difference between chaotic and stochastic processes. This difference may be explained by Oseledec's (1968) theorem, which guarantees the stability of $\hat{\lambda}_{max}$ in chaotic processes, and by the infinite dimensionality of the noise present in stochastic processes. This behaviour recalls the well-known process of saturation of the correlation dimension in a chaotic time series when the embedding dimension increases. As a matter of fact, this is the base of the test proposed by Grassberger and Procaccia (1983) to detect deterministic chaos.

4. A new test for distinguishing chaos from random behaviour via Lyapunov exponents

In this section, we propose a new test, based on the stability of the largest Lyapunov exponent from different sample sizes, to detect chaotic dynamics in time series. As we will see, this new test is rather powerful when compared to different stochastic alternatives, both linear and nonlinear.

This new test has a deterministic process as the null hypothesis, while the alternative hypothesis is that of a stochastic process (i. e., high-dimensional chaos), since randomness can be viewed as infinite-dimensional chaos.

In order to improve the estimation of largest Lyapunov exponents and following Hastie *et al.* (2001), we will use a procedure known as “bagging” that is based on averaging bootstrap samples. Bootstrap aggregation or bagging, averages the estimations over a collection of bootstrap samples, thereby reducing the variance. Bagging can dramatically reduce the variance of unstable procedures, leading to improved estimations because averaging reduces variance and leaves bias unchanged, which will often decrease mean squared error. Given this, assume a time series of length N , $\{x_1, x_2, \dots, x_N\}$. Let us divide the time series into different subsamples, each one of which contains the precedent $\{x_1, x_2, \dots, x_{T_1}, \dots, x_{T_2}, \dots, x_{T_{r-1}}, \dots, x_{T_r} = x_N\}$, and consider an empirical distribution of the largest Lyapunov exponent from 100 moving block bootstraps of this time series for the different subsamples $\{x_1, x_2, \dots, x_{T_i}\}$, for $i=1, \dots, r$. Therefore, our estimation of the dominant Lyapunov exponent for sample size T_i will be $\langle \lambda_{max}(T_i) \rangle$, that is, the mean of the distributions of the 100 largest Lyapunov exponents computed from those sample sizes T_i , which corresponds to the bootstrap aggregation or bagging of 100 bootstrap samples of largest Lyapunov exponents.

Given that we have shown in Figure 1 that the largest Lyapunov exponent stabilises (or even decreases, as for example in the case of Hénon and Lorenz attractors) when increasing the sample size in a deterministic process, but it increases with the sample size in a stochastic process, we propose using $\langle \lambda_{max}(T_i) \rangle$ to test for the stability of the largest Lyapunov exponent. This non-increasing property of largest Lyapunov exponent with the sample size for chaotic processes may be tested recalling the traditional econometric test of linear independence between the bootstrap aggregation of the 100 largest Lyapunov exponents $\langle \lambda_{max}(T) \rangle$, in every sample size, and the sample size T . To that end, we have performed a linear regression of

$$\langle \lambda_{max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T \quad \text{for } T = T_1, \dots, T_r = N, \quad (3)$$

so that the estimated parameter $\hat{\alpha}_1$ can be used to test if the largest Lyapunov exponent does not increase with sample size, implying an underlying deterministic process.

The null hypothesis H_0 and the alternative hypothesis H_1 are formulated as follows:

$$H_0 : \alpha_1 \leq 0 \quad (\text{deterministic process})$$

$$H_1 : \alpha_1 > 0 \quad (\text{stochastic process})$$

Observe that the alternative hypothesis of this test is closely associated with the estimation of λ_{max} through a direct method like Rosenstein's, where pure random processes have positive largest Lyapunov exponents (infinite in theory). Notice also that in order to implement this statistical test it is convenient to use the estimate of the asymptotic variance-covariance matrix of α_0 and α_1 proposed by Newey and West (1987) that is robust with respect to both heteroskedasticity and autocorrelation for the OLS estimations.

5. Applications

In this section we test for deterministic chaos using the simulated data from the seven models presented in the previous section. In all cases, the largest Lyapunov exponents were estimated using the algorithm proposed in Rosenstein *et al.* (1993).

Following Barnett *et al.* (1997), we compute our tests twice: for small samples of 380 observations and for large samples of 2000 observations. For the case of 380 observations, the subsample sizes are as follows:

$$T_1 = 200, T_2 = 220, T_3 = 240, T_4 = 260, T_5 = 280, \\ T_6 = 300, T_7 = 320, T_8 = 340, T_9 = 360, T_{10} = 380.$$

For the case of 2000 observations, the subsample sizes are as follows:

$$T_1 = 1000, T_2 = 1020, T_3 = 1040, \dots, T_{50} = 1980, T_{51} = 2000.$$

The key parameters in the Rosenstein *et al.*'s algorithm have been selected, both for deterministic and stochastic processes, following the author's recommendations, as follows:

- The embedding dimension, which coincides with the moving-block length d , has been selected between two and six.

- The lag or reconstruction delay has been fixed to one in all cases. In the case of Lorenz's series, this is due to an a priori sampling lag of $\tau = 10$ which has been used in order to avoid significant autocorrelations between data.
- The mean period of the time series, which restricts nearest neighbours to having a temporal separation greater than the mean period, allows us to consider each pair of neighbours satisfying this constraint as being sufficiently close together initial conditions for different trajectories. Following Rosenstein *et al.*'s recommendations and his program MTRCHAOS 1.0, we estimated the mean period as the reciprocal of the mean frequency of the power spectrum.

On the other hand, consistently locating the region for extracting $\lambda_{\max}$ without a priori knowledge of the correct slope in the linear region is a delicate question, as several authors have pointed out. After a short transition, there is a long linear region that is used to extract the largest Lyapunov exponent. Implementing our test, the number of discrete-time steps allowed for divergence between nearest neighbours has been set at $i = 3$. The location of the linear region to extract $\lambda_{\max}$ proposed by Rosenstein *et al.* is necessarily visual and therefore difficult to reproduce in the bootstrapping framework. In this case our selection for the number of discrete-time steps allowed for divergence between nearest neighbours may produce a small bias in the estimation of $\lambda_{\max}$ with respect to original Rosenstein *et al.*'s paper. Nevertheless, these biases do not appear to be very significant in the implementation of our test.

Tables 1 to 7 show the results of our regression testing for the stability of the mean largest Lyapunov exponent $\langle \lambda_{\max}(T) \rangle$ (for a sample of 100 largest Lyapunov exponents estimated by bootstrapping) when the sample size T increases.

As can be seen, if the 1% marginal significance level is used, for the Feigenbaum (Table 1) and Hénon (Table 6) series the test correctly distinguish deterministic from random behaviour for sample sizes of 380 and 2000 observations.

Table 1: Test for equality of the largest Lyapunov exponents from different sample sizes. Feigenbaum process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00005 (1.78217)	0.00000 (-0.36288)
Block size=3	0.00004 (0.96200)	0.00000 (1.35237)
Block size=4	0.00005 (1.39480)	0.00000 (1.56978)
Block size=5	0.00013 (0.45402)	0.00000 (1.48966)
Block size=6	0.00006 (1.67328)	0.00000 (0.13981)
Notes:		
(1) OLS estimation of the linear regression $\langle \lambda_{\max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.		
(2) ^a denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ (<i>deterministic process</i>) at the 1% level.		
(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.391).		

Table 2: Test for equality of the largest Lyapunov exponents from different sample sizes. GARCH process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00056 (23.62802 ^a)	0.00009 (13.20015 ^a)
Block size=3	0.00043 (10.71145 ^a)	0.00008 (23.45342 ^a)
Block size=4	0.00035 (9.22046 ^a)	0.00005 (19.20440 ^a)
Block size=5	0.00038 (8.27279 ^a)	0.00003 (10.79934 ^a)
Block size=6	0.00024 (13.45544 ^a)	0.00002 (18.13112 ^a)
Notes:		
(1) OLS estimation of the linear regression $\langle \lambda_{\max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.		
(2) ^a denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ (<i>deterministic process</i>) at the 1% level.		
(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.391).		

Table 3: Test for equality of the largest Lyapunov exponents from different sample sizes. NLMA process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00054 (5.51635 ^a)	0.00009 (29.46069 ^a)
Block size=3	0.00026 (5.22415 ^a)	0.00008 (47.99761 ^a)
Block size=4	0.00004 (2.75127)	0.00005 (36.26796 ^a)
Block size=5	0.00016 (8.63505 ^a)	0.00004 (13.31915 ^a)
Block size=6	0.00018 (6.73369 ^a)	0.00032 (13.19118 ^a)
Notes:		
(1) OLS estimation of the linear regression $\langle \lambda_{\max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.		
(2) ^a denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ (<i>deterministic process</i>) at the 1% level.		
(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.391).		

Table 4: Test for equality of the largest Lyapunov exponents from different sample sizes. ARCH process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00048 (20.62628 ^a)	0.00008 (32.82068 ^a)
Block size=3	0.00035 (7.38378 ^a)	0.00006 (19.62940 ^a)
Block size=4	0.00037 (6.96621 ^a)	0.00005 (20.18802 ^a)
Block size=5	0.00026 (5.64994 ^a)	0.00003 (20.62653 ^a)
Block size=6	0.00019 (9.03211 ^a)	0.00000 (24.22810 ^a)
Notes:		
(1) OLS estimation of the linear regression $\langle \lambda_{\max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.		
(2) ^a denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ (<i>deterministic process</i>) at the 1% level.		
(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).		

Table 5: Test for equality of the largest Lyapunov exponents from different sample sizes. ARMA process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00074 (6.28863 ^a)	0.00009 (27.21705 ^a)
Block size=3	0.00054 (5.73568 ^a)	0.00006 (27.24370 ^a)
Block size=4	0.00023 (3.07224 ^a)	0.00004 (22.57941 ^a)
Block size=5	0.00023 (3.79438 ^a)	0.00002 (12.79781 ^a)
Block size=6	0.00020 (4.62910 ^a)	0.00002 (13.30009 ^a)
Notes:		
(1) OLS estimation of the linear regression $\langle \lambda_{\max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.		
(2) ^a denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ (<i>deterministic process</i>) at the 1% level.		
(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).		

Table 6: Test for equality of the largest Lyapunov exponents from different sample sizes. Hénon process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00003 (1.63292)	-0.00001 (-9.49223)
Block size=3	-0.00004 (-1.17766)	-0.00001 (-4.46646)
Block size=4	0.00004 (0.98808)	-0.00001 (-5.33878)
Block size=5	-0.00001 (-0.17596)	-0.00001 (-6.7262)
Block size=6	-0.00003 (-0.74018)	-0.00001 (-0.67674)
Notes:		
(1) OLS estimation of the linear regression $\langle \lambda_{\max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.		
(2) ^a denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ (<i>deterministic process</i>) at the 1% level.		
(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).		

Table 7: Test for equality of the largest Lyapunov exponents from different sample sizes. Lorenz process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00018 (2.61055)	0.00001 (6.08057 ^a)
Block size=3	0.00011 (1.64782)	-0.00000 (-1.06341)
Block size=4	0.00005 (0.69311)	-0.00001 (-11.34195)
Block size=5	0.00008 (1.27820)	-0.00001 (-6.48720)
Block size=6	0.00015 (3.55523 ^a)	-0.00001 (-2.90258)
Notes:		
(1) OLS estimation of the linear regression $\langle \lambda_{\max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.		
(2) ^a denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ (<i>deterministic process</i>) at the 1% level.		
(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).		

Regarding the Lorenz attractor (Table 7), for T=380, the tests incorrectly reject the null hypothesis for small (d=2) and large (d=6) embedding dimensions. Two comments are in order. First, rejection of the null hypothesis for d=2 is not surprising for the three dimensional nature of Lorenz's attractor for which dynamics in two dimensions is poorly represented. Second, rejection of null for d=6 for small sample size can be explained by the well known phenomenon described as "the curse of dimensionality", a general problem that always manifests itself when local methods in high dimension are employed (Hastie *et al.*, 2001).

There exists an important observation with respect to the largest Lyapunov exponent estimations obtained in the Lorenz attractor (Table 7). The numbers that we show in Table 7 are not the largest Lyapunov exponents of the Lorenz attractor (for different block sizes) because of it is not possible to obtain these numbers without knowing the lag or reconstruction delay Δt used when sampling our time series from the continuous system. This lag Δt is crucial because the Lyapunov exponents measure the rate at which system processes create or destroy information per time units $i\Delta t$ in expressions (1) and (2). So exponents are expressed in bits of information per second. From this point of view the knowledge of the precise time units present in the time series is crucial in order to estimate the exact size of the Lyapunov exponents. Without this a priori information it is impossible to guess the precise time scale of the exponents obtained by Rosenstein et al algorithm.

In order to accomplish our test for discrete processes $\Delta t = 1$ is taken in equation (1) and the numbers calculated in Tables 1 to 6 are really estimations of the largest Lyapunov exponents for different values of the parameters. Nevertheless, the numbers obtained in Table 7 are only fractions of the largest Lyapunov exponents. Observe that in this case $\lambda_{\max}$ is estimated through a time series obtained by a fourth-order Runge-Kutta method, where the sampling period of integration is of the order $\Delta t \approx 0.01$. In addition, the time series used for estimating $\lambda_{\max}$ is obtained by sampling the integrated trajectory with a time lag $\tau = 10$. So the numbers of Table 7 have to be multiplied by the factor $\frac{1}{\Delta t \tau} \approx 10$ in order to get the real scale of the largest Lyapunov exponent in the Lorenz system.

Finally, note that the test correctly rejects the null hypothesis of determinism for all of the stochastic processes and for all sample sizes (see Tables 2 to 5) with the only exception being the NLMA stochastic model for sample size of 380, where the null hypothesis of deterministic process is not rejected.

Therefore, our simulation results suggest that our test correctly rejects chaos for the GARCH, NLMA, ARCH and ARMA stochastic processes in small and large sample sizes and for all embedding dimensions, except for the NLMA stochastic model, for 380 sample sizes. On the contrary, our test does not reject chaos in the case of the well-known chaotic processes for moving-block lengths from 3 to 5 for a sample size of 380, and from 3 to 6 for a sample size of 2000.

So our test works perfectly for large samples (2000 observations) although for small samples (380 observations) the test may present some imprecisions. Bearing in mind the stochastic nature of the bootstrapping procedure used to estimate the mean of largest Lyapunov exponents $\langle \lambda_{\max}(T) \rangle$ which represents the basis of our test, it appears to be convenient to obtain the size and power of our statistical test, for both sample sizes, in the different models considered above. To that end, we have generated the quantity $\langle \lambda_{\max}(T) \rangle$ one hundred times in order to repeat the test, observing the percentage of times that the null hypothesis is rejected when it is correct (deterministic processes), and the percentage of times that the null hypothesis is accepted when it is false (stochastic processes). The size and power of our test in the different models are shown in Table 8 for the sample size of 380, and Table 9-, for the sample size of 2000, for the nominal levels of significance of 90%, 95% and 99%.

Table 8: Size and power of largest Lyapunov exponent test: sample size = 380						
	Level	Block size 2	Block size 3	Block size 4	Block size 5	Block size 6
Feigenbaum $\Pr(\text{accept } H_0/\alpha_1 \leq 0)$	90%	0.3100	0.3400	0.1300	0.1700	0.1400
	95%	0.6000	0.5500	0.3300	0.3200	0.4200
	99%	0.9000	0.8600	0.7500	0.7000	0.7500
GARCH $\Pr(\text{reject } H_0/\alpha_1 > 0)$	90%	1.0000	1.0000	1.0000	1.0000	1.0000
	95%	1.0000	1.0000	1.0000	1.0000	1.0000
	99%	1.0000	1.0000	1.0000	1.0000	1.0000
NLMA $\Pr(\text{reject } H_0/\alpha_1 > 0)$	90%	1.0000	1.0000	0.7000	1.0000	1.0000
	95%	1.0000	1.0000	0.5000	1.0000	1.0000
	99%	1.0000	1.0000	0.1500	1.0000	0.9900
ARCH $\Pr(\text{reject } H_0/\alpha_1 > 0)$	90%	1.0000	1.0000	1.0000	1.0000	1.0000
	95%	1.0000	1.0000	1.0000	1.0000	1.0000
	99%	1.0000	1.0000	1.0000	1.0000	1.0000
ARMA $\Pr(\text{reject } H_0/\alpha_1 > 0)$	90%	1.0000	1.0000	1.0000	1.0000	1.0000
	95%	1.0000	1.0000	1.0000	1.0000	1.0000
	99%	1.0000	1.0000	0.2200	1.0000	1.0000
Hénon $\Pr(\text{accept } H_0/\alpha_1 \leq 0)$	90%	0.2200	1.0000	0.9400	0.9700	1.0000
	95%	0.7700	1.0000	0.9900	0.9900	1.0000
	99%	1.0000	1.0000	1.0000	1.0000	1.0000
Lorenz $\Pr(\text{accept } H_0/\alpha_1 \leq 0)$	90%	0.0000	0.0000	1.0000	0.1500	0.0000
	95%	0.0000	0.0100	1.0000	0.4800	0.0000
	99%	0.6400	0.5600	1.0000	0.9100	0.1000

Table 9: Size and power of largest Lyapunov exponent test: sample size = 2000						
	Level	Block size 2	Block size 3	Block size 4	Block size 5	Block size 6
Feigenbaum $\Pr(\text{accept } H_0/\alpha_1 \leq 0)$	90%	0.8700	0.9300	0.7100	0.7200	0.7500
	95%	0.9400	0.9800	0.8700	0.8000	0.8600
	99%	0.9800	0.9900	0.9500	0.9500	0.9700
GARCH $\Pr(\text{reject } H_0/\alpha_1 > 0)$	90%	1.0000	1.0000	1.0000	1.0000	1.0000
	95%	1.0000	1.0000	1.0000	1.0000	1.0000
	99%	1.0000	1.0000	1.0000	1.0000	1.0000
NLMA $\Pr(\text{reject } H_0/\alpha_1 > 0)$	90%	1.0000	1.0000	1.0000	1.0000	1.0000
	95%	1.0000	1.0000	1.0000	1.0000	1.0000
	99%	1.0000	1.0000	1.0000	1.0000	1.0000
ARCH $\Pr(\text{reject } H_0/\alpha_1 > 0)$	90%	1.0000	1.0000	1.0000	1.0000	1.0000
	95%	1.0000	1.0000	1.0000	1.0000	1.0000
	99%	1.0000	1.0000	1.0000	1.0000	1.0000
ARMA $\Pr(\text{reject } H_0/\alpha_1 > 0)$	90%	1.0000	1.0000	1.0000	1.0000	1.0000
	95%	1.0000	1.0000	1.0000	1.0000	1.0000
	99%	1.0000	1.0000	1.0000	1.0000	1.0000
Hénon $\Pr(\text{accept } H_0/\alpha_1 \leq 0)$	90%	1.0000	1.0000	1.0000	1.0000	1.0000
	95%	1.0000	1.0000	1.0000	1.0000	1.0000
	99%	1.0000	1.0000	1.0000	1.0000	1.0000
Lorenz $\Pr(\text{accept } H_0/\alpha_1 \leq 0)$	90%	0.0000	1.0000	1.0000	1.0000	1.0000
	95%	0.0000	1.0000	1.0000	1.0000	1.0000
	99%	0.0000	1.0000	1.0000	1.0000	1.0000

As we can see in Table 8, with a sample size of 380 the power of our test against stochastic processes is 1 for the entire stochastic model analysed, and for most moving block sizes. There are only two exceptions: the NLMA model that fails on a block size of 4 for all levels, and the ARMA model that also fails also on a block size of 4, but only at 99% level. In order to construct Table 8 we have considered the first 380 observations of the available series. Nevertheless if we consider as in NLMA as in ARMA models the second subsample of 380 observations of the whole series, the power is again 1.

In Table 8 we also have estimated the size of our test for a sample size of 380. The results are diverse and depend on the process and the block size. While in the Hénon series the size of the tests are acceptable for most levels and block sizes, for the Lorenz and Feigenbaum models the results are poorer.

Table 9 shows the size and power of our test for the sample size of 2000. In this case our test emphatically rejects the stochastic processes and does not adequately reject the chaotic processes, with the only exception of Lorenz series for a block size of 2 where the chaotic dynamics is modelled in a state space whose dimension is lower than that of the attractor, so that only a projection of the dynamic remains, and determinism is invisible and the dynamics look random.

6. Comparison of the power performance of the new test with Bask and Gençay's (1998)

Bask and Gençay (1998) proposed using a moving-block bootstrap procedure to test for the presence of a positive Lyapunov exponent in an observed stochastic time series. The null hypothesis H_0 and the alternative hypothesis H_1 are formulated as follows:

$$H_0 : \lambda_{\max} = 0 \text{ (no chaotic process)}$$

$$H_1 : \lambda_{\max} > 0 \text{ (chaotic process)}$$

where $\lambda_{\max}$ is the largest Lyapunov exponent.

The test scheme consists of the following steps:

- i) Reconstruct the phase space of the time series $\{x_1, x_2, \dots, x_N\}$ with an embedding dimension d and estimate the largest Lyapunov exponent $\lambda_{\max}$ using any existing algorithm. Each d -history of the reconstructed phase space will be considered as a block, obtaining in this way a sequence of blocks $\{B_1^d, \dots, B_{N-d+1}^d\}$.

- ii) Re-sample, with replacement, k blocks of the reconstructed phase space, where $k = \text{int}(N/d)$. The subfamily of blocks $\{B_{i_1}^d, \dots, B_{i_k}^d\}$ constitutes the bootstrap sample.
- iii) From this subfamily, estimate the largest Lyapunov exponent $\tilde{\lambda}_{max}$ for the time series under study and calculate $\tilde{\lambda}_{max} - \hat{\lambda}_{max}$.
- iv) Repeat steps ii)-iii) a large number of times to construct an empirical distribution of $\tilde{\lambda}_{max} - \hat{\lambda}_{max}$.
- v) Construct a one-sided 97.5% confidence interval by calculating the critical value as $\hat{\lambda}_{max} - q(97.5\%)$, following from $Pr\{\tilde{\lambda}_{max} - \hat{\lambda}_{max} < q(97.5\%)\} = 0.975$, where $q(97.5\%)$ is the quartile of the distribution in step iv).
- vi) If $\hat{\lambda}_{max} - q(97.5\%) > 0$, then the null hypothesis is rejected, which means that the dynamics are chaotic.

Tables 10 to 16 show the results of applying the test proposed by Bask and Gençay (1998) to the simulated data examined in the previous section for moving blocks of different sizes (i. e., $d=2$ to 6) and for different sample sizes ($T=380$ and 2000).

Table 10: Results of Bask and Gençay (1998)'s test.

Feigenbaum process.

Sample size	T=380 (1)				T=2000 (1)			
	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$
Block size=2	0.01245	-0.04361	-0.03196	-0.01894	0.01605	-0.00611	0.00452	0.00533
Block size=3	0.01256	-0.04639	-0.03838	-0.02952	0.01608	-0.00288	0.00013	0.00151
Block size=4	0.01064	-0.04087	-0.02557	-0.02545	0.01572	-0.00465	-0.00364	0.00174
Block size=5	0.01029	-0.05308	-0.04173	-0.03743	0.01566	-0.02307	-0.00535	0.00053
Block size=6	0.01071	-0.04589	-0.03008	-0.02339	0.01574	-0.00955	-0.00131	-0.00013

Note:

(1) $\hat{\lambda}_1$ is an estimation of the largest Lyapunov exponent, and $q(\cdot)$ is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$.

$\tilde{\lambda}_1$ is the largest exponent from the bootstrap sample. The number of bootstrap values is 100.

Table 11: Results of Bask and Gençay (1998)'s test.

GARCH process.

Sample size	T=380 (1)				T=2000 (1)			
	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$
Block size=2	0.90248	0.86682	0.87272	0.87448	1.13738	1.12235	1.12365	1.12809
Block size=3	0.65089	0.60719	0.62409	0.62675	0.79050	0.76752	0.77342	0.77685
Block size=4	0.47382	0.42162	0.44309	0.44555	0.56657	0.54499	0.54708	0.54941
Block size=5	0.34417	0.28803	0.30950	0.31230	0.41925	0.39863	0.40461	0.40599
Block size=6	0.25150	0.20536	0.21890	0.22511	0.32780	0.30987	0.31208	0.31348

Note:

(1) $\hat{\lambda}_1$ is an estimation of the largest Lyapunov exponent, and $q(\cdot)$ is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$.

$\tilde{\lambda}_1$ is the largest exponent from the bootstrap sample. The number of bootstrap values is 100.

Table 12: Results of Bask and Gençay (1998)'s test.

NLMA process.

Sample size	T=380 (1)				T=2000 (1)			
	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$
Block size=2	0.92826	0.87644	0.89129	0.89253	1.15676	1.13687	1.14444	1.14546
Block size=3	0.62846	0.58843	0.58900	0.59318	0.80085	0.77880	0.78517	0.78652
Block size=4	0.42392	0.37118	0.38158	0.38454	0.57138	0.55181	0.55390	0.55710
Block size=5	0.33293	0.29567	0.30253	0.30510	0.42726	0.40485	0.40707	0.41118
Block size=6	0.26197	0.21052	0.22431	0.22692	0.33006	0.31156	0.31368	0.31728

Note:

(1) $\hat{\lambda}_1$ is an estimation of the largest Lyapunov exponent, and $q(\cdot)$ is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$.

$\tilde{\lambda}_1$ is the largest exponent from the bootstrap sample. The number of bootstrap values is 100.

Table 13: Results of Bask and Gençay (1998)'s test.

ARCH process.

Sample size	T=380 (1)				T=2000 (1)			
	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$
Block size=2	0.90849	0.86379	0.87563	0.87910	1.15356	1.13192	1.13369	1.13932
Block size=3	0.64325	0.59138	0.60749	0.61349	0.80730	0.78453	0.78996	0.79239
Block size=4	0.44814	0.39919	0.40002	0.40351	0.57090	0.54591	0.55642	0.55699
Block size=5	0.33330	0.28696	0.29223	0.29676	0.42729	0.40256	0.40865	0.41026
Block size=6	0.24838	0.18943	0.20549	0.21688	0.33263	0.31034	0.31413	0.31450

Note:

(1) $\hat{\lambda}_1$ is an estimation of the largest Lyapunov exponent, and $q(\cdot)$ is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$.

$\tilde{\lambda}_1$ is the largest exponent from the bootstrap sample. The number of bootstrap values is 100.

Table 15: Results of Bask and Gençay (1998)'s test.

Hénon process

Sample size	T=380 (1)				T=2000 (1)			
	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$
Block size=2	0.41333	0.38076	0.38431	0.38687	0.38359	0.36296	0.36611	0.36943
Block size=3	0.38982	0.34653	0.35916	0.36146	0.36573	0.34091	0.34473	0.34950
Block size=4	0.37672	0.32673	0.32948	0.33680	0.35342	0.32532	0.32808	0.33549
Block size=5	0.36801	0.31584	0.32989	0.33257	0.35118	0.31993	0.32857	0.33148
Block size=6	0.35042	0.27605	0.28109	0.29070	0.35058	0.31773	0.32681	0.32916

Note:

(1) $\hat{\lambda}_1$ is an estimation of the largest Lyapunov exponent, and $q(\cdot)$ is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$.

$\tilde{\lambda}_1$ is the largest exponent from the bootstrap sample. The number of bootstrap values is 100.

Table 16: Results of Bask and Gençay (1998)'s test.

Lorenz process.

Sample size	T=380 (1)				T=2000 (1)			
	$\hat{\lambda}_l$	Critical value $\hat{\lambda}_l - q(99\%)$	Critical value $\hat{\lambda}_l - q(97.5\%)$	Critical value $\hat{\lambda}_l - q(95\%)$	$\hat{\lambda}_l$	Critical value $\hat{\lambda}_l - q(99\%)$	Critical value $\hat{\lambda}_l - q(97.5\%)$	Critical value $\hat{\lambda}_l - q(95\%)$
Block size=2	0.48765	0.44554	0.45900	0.46064	0.51460	0.48868	0.49413	0.49462
Block size=3	0.39440	0.31527	0.33911	0.34908	0.36864	0.33994	0.34279	0.34602
Block size=4	0.31940	0.24911	0.25131	0.27093	0.30247	0.26868	0.27266	0.27592
Block size=5	0.26097	0.21978	0.22776	0.23286	0.25042	0.21438	0.22553	0.23011
Block size=6	0.21320	0.12033	0.13324	0.15345	0.21489	0.17904	0.18834	0.19051

Note:

(1) $\hat{\lambda}_l$ is an estimation of the largest Lyapunov exponent, and $q(\cdot)$ is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_l - \hat{\lambda}_l$.

$\tilde{\lambda}_l$ is the largest exponent from the bootstrap sample. The number of bootstrap values is 100.

As can be seen, Bask and Gençay's test incorrectly rejects the null hypothesis $\lambda_{max} = 0$, by not rejecting the alternative hypothesis $\lambda_{max} > 0$ of chaos in any of the stochastic processes considered in this paper (GARCH, NLMA, ARCH and ARMA) in both sample sizes. In contrast, our test does correctly reject the null hypothesis, by not rejecting the existence of chaos for the series of Hénon and Lorenz in both sample sizes. On the other hand, our test does not reject the null hypothesis of a non-chaotic process for the Feigenbaum series for all block sizes and significance levels for a sample size of 380, while the test fails to recognise chaos in the most cases for a sample size of 2000.

7. Concluding remarks

Empirical research on the detection of chaotic behaviour has expanded rapidly, but results on small sample sizes have tended to be rather inconclusive, due to the lack of appropriate testing methods.

The problem of distinguishing chaotic from random behaviour is a very complex task limited by the number of observations available and the dimension of the chaotic attractor that generates the process. If the dimension of the attractor is large enough, the amount of data needed to test it may be prohibitive.

The general practice has been to take the existence of a positive Lyapunov exponent as an indication that the system is chaotic. However, this condition is not sufficient for the detection of chaos, and does not help us to distinguish a chaotic process from stochastic one. Indeed, any standard direct algorithm for calculating the largest Lyapunov exponent will find a finite, positive value for this exponent, both for chaotic and stochastic processes.

In this paper, we combine the bootstrap statistical framework for hypothesis testing using the computed Lyapunov exponents (Gençay, 1996), with the ergodic theory of deterministic dynamical systems in order to develop a new test to detect chaotic dynamics in time series. The new test is based on the stability of the mean in the distributions of the largest Lyapunov exponent estimated from different sample sizes, which is guaranteed by Oseledec's (1968) theorem. This theorem provides a strong feature of deterministic processes that is not shared by stochastic processes. We show that, while for stochastic processes (both linear and non-linear) the largest Lyapunov exponent increases with the sample size, for chaotic series the largest Lyapunov exponent is invariant when increasing the sample size. We compute the largest Lyapunov exponent using a robust version of the algorithm proposed by Rosenstein *et al.*

(1993), considering the mean of divergences between pairs of neighbouring trajectories.

We have applied this new test to the simulated data used in the single-blind controlled competition among tests for nonlinearity and chaos generated by Barnett *et al.* (1997), as well as several chaotic series, both for small and large samples (380 and 2000 observations, respectively). The results suggest that the new test has a high discriminatory power against interesting stochastic alternatives, both linear and nonlinear (GARCH, NLMA, ARCH and ARMA).

Indeed, the new test proposed here is able to correctly reject chaotic behaviour for all stochastic process considered (GARCH, NLMA, ARCH and ARMA) for sample sizes of 2000. Meanwhile for sample sizes of 380 chaotic behaviour is also rejected with the only exception of NLMA for moving-block lengths of 4. Furthermore, chaotic behaviour is always correctly identified for all the chaotic processes considered for sample sizes of 2000 (with the only exception of Lorenz series for block size $d=2$), while for sample sizes of 380 chaotic behaviour is always correctly accepted for moving-block lengths from 3 to 5. However, for extreme values of the block size (2 and 6) our test incorrectly rejects the null hypothesis for a sample size of 380 for the Lorenz process. For sample sizes of 2000 observations, our test does not correctly reject the null for all chaotic processes in every block size considered, except for the Lorenz series in block size 2, which is acceptable due to the three dimensional nature of this attractor.

Studying the size and power for sample sizes of 2000 we observe that our test is very powerful when compared to stochastic alternatives for all block sizes, and has size equal to 1 for all chaotic processes with the exception of Lorenz process for a block size of 2, where the dynamics are not reconstructed. For the sample size of 380 our test is also very powerful when compared to different stochastic alternatives (both linear and nonlinear). Nevertheless the size of the test diminishes for the chaotic processes, being inferior to the nominal size (with the exception of Hénon series).

When comparing the results from our test with those from the competing test proposed by Bask and Gençay (1998), we conclude that the latter cannot correctly distinguish between chaotic and stochastic processes, although it correctly rejects the null hypothesis (of no chaos) for two of the three chaotic processes considered, what is not a surprise given some well-known properties of such processes.

Therefore, the results presented in this paper suggest that our test provides an improvement over several tests available in the literature, since it

has the ability to distinguish between deterministic or stochastic processes, and it proves to be very powerful when compared to different stochastic alternatives (both linear and nonlinear) for both large and small sample sizes.

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