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Efficiency and Endogenous Fertility

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NON-TECHNICAL SUMMARY

This paper explores the properties of several notions of efficiency (A-efficiency, P-efficiency and Millian efficiency) to evaluate allocations in a general overlapping generations setting with endogenous fertility and descendant altruism. The setting includes, as particular cases, environments with infinite horizon, dynastic altruism *à la* Barro and Becker (1988) as well as environments with infinite horizon or other forms of non-dynastic altruism.

More specifically, we focus on a two-period, overlapping generations setting in which *a)* the set of fertility choices is an unbounded interval in the real line; *b)* any two agents of the same generation who get to be alive have the same labor endowment and the same utility function, that depends on their consumption of a homogeneous good, on the number of children they decide to bear and on the utility obtained by their descendants.

Initially, we focus on the notion of A-efficiency -proposed by Golosov, Jones and Tertilt (*Econometrica*, 2007)- and raise some normative concerns: in many environments, the set of symmetric, interior, A-efficient allocations is empty. We show that every symmetric, A-efficient allocation must maximize the utility of the dynasty head among feasible (but not necessarily symmetric) allocations. A straightforward implication of this result is that, in many environments, achieving efficiency without discriminating or exploiting some of the agents is impossible. We regard these results as reminiscent of the so called *Repugnant Conclusion* arising in the Social Choice literature.

To overcome this problem, we then propose to evaluate the efficiency of a given allocation with a particular specification of P-efficiency -proposed also by Golosov *et al.* (2007)-, for which the utility attributed to the unborn depends on the utility level achieved by those who get to be born in a given allocation. Millian efficient allocations can be described as P-efficient allocations for which P-efficiency holds for a wide range of specifications of the utility attributed to the unborn.

To conclude the paper, we explore the efficiency properties of a decentralized mechanism in which the agents, endowed with well-defined property rights over the commodities available in the economy, are free to trade these rights (or transfer them) to pursue their own interests. With this purpose, we slightly modify the notion of competitive equilibrium proposed by Golosov *et al.* (2007) and impose that transfers from parents to their children are non-negative.

We show that if the (constrained) value functions associated to the notion of competitive equilibrium are concave on a certain range, then there exist competitive equilibria that are interior and symmetric and, hence, potentially A-inefficient. In view of our previous results, when altruism is non-dynastic, symmetric equilibria are always A-inefficient, while if altruism is dynastic, symmetric equilibria may be A-inefficient if either *a)* some of the (unconstrained) value functions associated to A-efficiency are non-concave, or *b)* some of the non-negativity constraints on gifts or bequests are binding. In our view, this fact seeds

some doubts on the notion of A-efficiency as a useful concept to predict the behavior of the agents in real economies, as the notion of Pareto efficiency does in the Edgeworth-Coase tradition.

For the notion of Millian efficiency (and, hence, for P-efficiency), however, a version of the First Welfare Theorem holds. To be more precise, we show that *a)* a symmetric competitive equilibrium is a -statically- Millian efficient allocation; and *b)* if long-run wages do not exceed the capitalized costs of rearing children, then a competitive equilibrium is also -dynamically- Millian efficient.

Efficiency and Endogenous Fertility*

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Abstract

This paper explores the properties of several notions of efficiency ($\mathcal{A}-$efficiency, $\mathcal{P}-$efficiency and Millian efficiency) to evaluate allocations in a general overlapping generations setting with endogenous fertility and descendant altruism that includes, as a particular case, Barro and Becker's (1988) model of fertility choice. We first focus on the notion of $\mathcal{A}-$efficiency, proposed by Golosov, Jones and Tertilt (*Econometrica*, 2007) and show that, in many environments, the set of symmetric, interior, $\mathcal{A}-$efficient allocations is empty. To overcome this problem, we then propose to evaluate the efficiency of a given allocation with a particular specification of $\mathcal{P}-$efficiency –proposed also by Golosov *et al.*– for which the utility attributed to the unborn depends on the utility level achieved by those who get to be born in a given allocation. For a large class of specifications of the function determining the utility attributed to the unborn, every Millian efficient allocation, that is, every symmetric allocation that is not $\mathcal{A}-$dominated by any other symmetric allocation, is also $\mathcal{P}$-efficient. Finally, we restate the First Welfare Theorem by showing that *a*) every competitive equilibrium is a –statically– Millian efficient allocation; and that *b*) if long-run wages do not exceed the capitalized costs of rearing children, then competitive equilibria are also –dynamically– Millian efficient.

Key words: Efficiency, Optimal Population, Endogenous Fertility, $\mathcal{A}-$efficiency, $\mathcal{P}-$efficiency, Millian efficiency

JEL: D91, H21, H5, E62, J13

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1 Introduction

This paper explores the properties of the notions of $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency –proposed by Golosov, Jones and Tertilt (*Econometrica*, 2007)–, as well as the notion of Millian efficiency –proposed by Conde-Ruiz, Giménez and Pérez-Nievas (*Review of Economic Studies*, 2010)– to evaluate allocations in a general overlapping generations setting with endogenous fertility and descendant altruism. The setting includes, as particular cases, environments with infinite horizon, dynastic altruism *à la* Barro and Becker (1988) as well as environments with finite horizon or other forms of non-dynastic altruism.

The most commonly used optimality notion in standard normative economic analysis is that of Pareto efficiency. This notion of efficiency relies on the well-known Pareto criterion to compare social alternatives, a criterion that allows one to construct a *partial ordering* on a set of alternatives from the *complete preference orderings* (defined on this set) of a *fixed* group of agents. An efficient allocation can be described as a maximal element of the partial order induced by the Pareto criterion on the set of feasible allocations.

With endogenous populations, one can still use the Pareto criterion to rank feasible allocations using the *partial orderings* of all potential agents, represented by the utility functions of the *living* agents. That is, an allocation can still be ranked as Pareto superior to another one if it is unanimously preferred by all potential agents according to their partial preference ordering. However, this implies that any two allocations with different population size cannot be ranked, since there is no way to know whether or not an agent who lives in one allocation a but not in other allocation a' is better off in the latter than he is in the former. To avoid this problem and preserve the partial order induced by the Pareto criterion, one needs to extend it to compare also allocations of different population size.

Although the issue seems to concern policymakers everywhere, the theoretical foundations of many proposals to alter fertility rates are rather weak, and we do not have enough theoretical grounds to determine what the optimal growth of the population is and, consequently, to claim that individual fertility choices lead to a too low or a too high population growth. Most of the literature has simply turned to identify *optimal allocations* with the solutions to alternative social welfare maximization problems: a first approach identifies socially optimal allocations with *steady state optimal allocations* (also referred to as *golden rule* allocations), that is, allocations that maximize the utility obtained by a representative consumer among those feasible stationary allocations;[1] while a second approach identifies optimal allocations with those maximizing a certain class of social welfare maximization problems, referred to as *Millian* or *Benthamite* depending on whether or not the welfare weight given to a generation in the social welfare function depends on the size of that generation.[2] Neither one of these two approaches has taken explicitly into account the fact that the Pareto criterion is not directly applicable to environments in which the set of agents is endogenous. Besides, a social welfare maximization problem may display two problems: it may not be well-defined and, if so, its solution identifies a unique "optimal" allocation which might not be enough for welfare comparisons of any two alternative allocations.

A different approach has emerged in the Social Choice literature.[3] There, the objective has been to derive, using different axioms, population principles that extend fixed population social

[1] See e.g., Samuelson (1975, 1976), Deardoff (1976), Eckstein and Wolpin (1985), Bental (1989), Michel and Pestieau (1993) and Jaeger and Kuhle (2009).

[2] See e.g., Nerlove, Razin and Sadka (1982, 1985), Cigno (1992), Groezen, Leers and Medjam (2003) or Razin and Sadka (1995, Ch.5) for a survey. More recent contributions are those by Renzo and Spataro (2011), Fabri, Boucekkine and Gozzi (2012), and Bucekkine and Fabri (2013).

[3] See e.g., Blackorby, Bossert and Donaldson (1995, 2002, 2005)

goodness relations so that they can rank alternatives with different populations. All papers in this line of research adopt a welfarist approach; that is, they rank alternatives with different population size using information about all those who ever live together with their levels of lifetime utility, and derive complete preference orderings on the set of alternatives that are representable by social welfare functions. As for fixed population environments, deriving a social welfare function might be too demanding if one is simply interested on a weak criterium to compare allocations –such as the Pareto criterion. Besides, it typically requires the assumption of interpersonal comparisons of utility, which is not required for the notion of Pareto efficiency.

Unlike these literature, some recent papers, among which we should mention Schweizer (1997), Lang (2005), Michele and Wigniolle (2007), and Conde-Ruiz *et al.* (2010) have provided normative principles to evaluate population policies in the context of an overlapping generations framework without altruism. However, these papers restrict their analysis to symmetric allocations, that is, to allocations in which any two agents of the same generation obtain the same consumption bundle. They all focus on an extension of the Pareto criterion (which is still referred to as (modified) Pareto dominance by Schweizer, Lang and Michele *et al.* and as $\mathcal{A}-$*dominance* by the latter) that compares any two allocations of different population size by comparing exclusively the welfare profiles of those agents who are alive in the two allocations. Given the symmetry restriction on the set of allocations that are comparable using the $\mathcal{A}-$dominance criterium, a feasible symmetric allocation is said to be efficient (or, using the term proposed by Conde-Ruiz *et al*, *Millian efficient*) if there does not exist an alternative feasible, symmetric allocation that provides all members of a generation with higher utility without decreasing the utility obtained by any other generation.[4] The name "Millian" refers to the fact that it is a notion of efficiency that generalizes the notion of Millian optimality mentioned above.

But the restriction to symmetric allocations seems too demanding. To fill this gap, Golosov, Jones and Tertilt (2007) have proposed two extensions of the Pareto criterion to compare allocations with different population size. The first of these extensions is referred to as the $\mathcal{A}-$dominance criterion and, as mentioned above, ranks any two allocations by applying the Pareto criterion using information of the preference profiles of those agents who are alive in the two allocations, without restricting welfare comparisons to symmetric allocations. The second extension, referred to as the $\mathcal{P}-$dominance criterion, is constructed from a previous assumption on the utility level obtained by potential non-born agents. This assumption, together with the existence of well-defined utility functions that represent the preferences of all potential agents in those allocations in which they are alive, provides a complete description of preferences of all potential agents in the economy across all allocations, which allows the authors to rank any two allocations of different population size by comparing (using the Pareto criterion and the utility level attributed to the unborn) the utility profiles of all potential agents in the two allocations. These two extensions of the notion of Pareto dominance give rise to two notions of efficiency, respectively referred to as $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency, to evaluate allocations in environments in which fertility decisions are endogenous.

Golosov, Jones and Tertilt (henceforth GJT) provide partial characterizations of the two notions of efficiency as the solutions to welfare maximization problems, and prove that, under relatively mild assumptions, $\mathcal{A}-$efficient allocations are either $\mathcal{P}-$efficient or are arbitrarily close to allocations that are also $\mathcal{P}-$efficient(see GJT 2007, Sec.4.3 Result 3). Thus, the $\mathcal{P}-$efficiency of $\mathcal{A}-$efficient allocations is robust to different specifications of the utility levels attributed to the unborn. In a framework with dynastic altruism *à la* Barro and Becker (1989), GJT also explore

[4]Recent applications –under different names– of this notion of efficiency in applied work, can be seen in Michel and Wigniolle (2010) and Oguro *et al* (2013) –who use the name "Representative Consumer Efficiency," and also in Fanti and Gori (2012a, 2012b), who use the name A-efficiency. .

the properties of a notion of equilibrium which results from the combination of the notion of competitive equilibria and the notion of subgame perfect equilibria of a voluntary transfer game played within families. In this environment, they also provide a version of the First Welfare Theorem by showing that such non cooperative equilibria are both $\mathcal{A}-$efficient and $\mathcal{P}-$efficient (see GJT 2007, Ths.1 and 2).

In this paper, we consider the properties of different notions of efficiency in a framework slightly more general than GJT's extension of Barro and Becker's model. We focus on a two-period, overlapping generations setting in which *a*) the set of fertility choices is an unbounded interval in the real line; *b*) any two agents of the same generation who get to be alive have the same labor endowment and the same utility function, that depends on their consumption of a homogeneous good, on the number of children they decide to bear and on the utility obtained by their descendants. The setting covers, as particular cases, a wide range of positive models of fertility choice, including models with dynastic altruism as well as models with weaker, or non-dynastic, forms of altruism. In this context, we show that the notion of $\mathcal{A}-$efficiency proposed by GJT face some important difficulties.

$\mathcal{A}-$Efficiency: normative concerns

Although the notion of Pareto efficiency does not incorporate distributional concerns, it is, in general, compatible with the existence of individual rights or with certain weak principles of horizontal equity. For example, in exchange economies, achieving Pareto efficiency does not impede that the agents are entitled with property rights on the commodities that are traded, because there are many efficient allocations that are compatible with these rights. Thus, excluding, as *socially undesirable*, those allocations in which some of the agents exploit others by appropriating their entire endowment (which are Pareto efficient) does not bring with it efficiency losses, because there are many other efficient allocations that compatible with these rights. Analogously, there are many Pareto efficient allocations for which any two identical agents (that is, agents with the same preferences and endowments) are treated symmetrically, and it is generally true that if a symmetric allocation is not dominated by any other symmetric allocation, then it is not dominated by any other (symmetric or non symmetric) allocation either. Thus, restricting the set of socially *desirable* outcomes to the set of symmetric allocations does not bring with it a welfare loss in the sense given by the Pareto criterion.

Our main concern with $\mathcal{A}-$efficiency is that, in many environments, this notion of efficiency is incompatible with individual rights or with seemingly weak principles of horizontal equity. To make this clearer, we explore the properties of the set of allocations that are $\mathcal{A}-$efficient and symmetric –which, in our setting, means that all agents of the same generation obtain the same consumption bundle. In Proposition 5, we show that every symmetric, $\mathcal{A}-$efficient allocation must maximize the utility of the dynasty head among feasible (but not necessarily symmetric) allocations. A straightforward implication of this result is that, in many environments, achieving efficiency without discriminating or exploiting some of the agents is impossible. More precisely:

- With finite-horizon altruism, that is, when parents care only about consumption decisions of their immediate descendants, the set of interior $\mathcal{A}-$efficient allocations is empty (Proposition 6). That is, achieving $\mathcal{A}-$efficient requires that some of the agents devote their entire environment -or labor capacity- to provide with resources to their parents. Since, in the context of the setting explored in the paper, the agents do not care on the welfare obtained by their parents, we regard this property as a form of exploitation. Observe also that if all the agents of a generation devote all their resources to finance their parents' consumption and do not have enough resources to finance the costs of rearing new children, the economy collapses in finite time.

- With infinite horizon altruism, the set of (interior and non-interior) symmetric allocations might also be empty (Proposition 7). This occurs in economies for which value functions are not concave. Since the set of feasible allocations arising in models with endogenous fertility is non convex, value functions might be non-concave in standard economies, as we show through a class of examples in the paper.

We regard these results as reminiscent of the so called *Repugnant Conclusion* (Parfit 1976) arising in the Social Choice literature (see, e.g., Blackorby, Bossert and Donaldson 1995, 2002, 2005), that applies to social welfare functions that rank allocations with a large number of people obtaining low utility higher than allocations with few people and high levels of utility. The intuition of why the notion of $\mathcal{A}-$efficiency reduces to dynastic maximization, which drives these results, is the following. Even in regular economies for which the dynasty head would not be willing to discriminate among their descendants if she could freely choose the amount of resources available to each of these descendants, she might be interested in discriminating among these descendants if she is forced to provide each of them with more resources than she would like to. In such cases, starting from an allocation a, it is always possible to find an allocation a' with more individuals that makes all people living under both a and a' better off than they were under a. Welfare improvements of this type (in the sense given by the $\mathcal{A}-$dominance criterion) can be achieved by enforcing every newcomer –that is, every individual living under a' who was not born under a– to receive fewer resources than those received by their siblings, so that parents are willing to afford the additional costs that increasing the size of the population might bring with it.

Avoiding repugnant consequences of A−efficiency: Millian efficiency as Robust P−efficiency

Motivated by our previous results, we explore whether or not the notion of $\mathcal{P}-$efficiency proposed by GJT is able to overcome the difficulties described above. As we argue in the paper, the answer depends on the function determining the utility level attributed to the unborn. If the utility attributed to the unborn is a constant critical level $\overline{u}$ (as GJT assume in most of their applications) which is always lower than the utility obtained by any alive agent, then the notion of $\mathcal{P}-$efficiency faces the same shortcomings faced by $\mathcal{A}-$efficiency. As $\overline{u}$ increases, the set of $\mathcal{P}-$efficient allocations becomes larger and includes all Millian efficient allocations that provide the living agents with utility levels below the critical level $\overline{u}$. In our view, the way by which the notion of $\mathcal{P}-$efficiency (with a constant critical level) mitigates the problems raised above is not entirely satisfactory because determining whether or not an allocation is $\mathcal{P}-$efficient becomes heavily dependent on the critical level $\overline{u}$, a cardinal value that, by definition, we cannot observe by watching the behavior of the agents.

However, other specifications of the utility attributed to the unborn may reduce this dependency, as long as the utility attributed to the unborn is a function of the utility achieved by those alive at any allocation. To be more precise, if the utility attributed to the unborn is a monotonic, symmetric function of the utility obtained by the living agents of the same generation, then every Millian efficient can be regarded as $\mathcal{P}-$efficient (Proposition 8). Thus, the set of Millian efficient allocations is a proper subset of the set of $\mathcal{P}-$efficient allocations or, put it in other words, Millian efficient allocation can arise as efficient (in this case, $\mathcal{P}-$efficient) even if the set of allocations that can be compared using the $\mathcal{P}-$dominance criterion is not restricted *a priori*. Furthermore, the $\mathcal{P}-$efficiency of Millian efficient allocations holds for a wide range of specifications of the utility attributed to the unborn. Assuming the utility attributed to the unborn is a monotonic, symmetric function of the utility obtained by the living agents of the same generation avoids cardinalist assessments in welfare comparisons.

Equilibrium Behavior: A Failure in The First Welfare Theorem?

To conclude the paper, we explore the efficiency properties of a decentralized mechanism in which the agents, endowed with well-defined property rights over the commodities available in the economy, are free to trade these rights (or transfer them) to pursue their own interests. With this purpose, we slightly modify the notion of competitive equilibrium proposed by GJT and impose that transfers from parents to their children are non-negative (so that they are voluntarily accepted by the latter).

First, we show that if the (constrained) value functions associated to the notion of competitive equilibrium are concave on a certain range, then there exist competitive equilibria that are interior and symmetric and, hence, potentially $\mathcal{A}-$inefficient. To be more precise, when altruism is non-dynastic, symmetric equilibria are always $\mathcal{A}-$inefficient, while if altruism is dynastic, symmetric equilibria may be $\mathcal{A}-$inefficient if either *a*) some of the (unconstrained) value functions associated to $\mathcal{A}-$efficiency are non-concave, or *b*) some of the non-negativity constraints on gifts or bequests are binding. In our view, this fact seeds some doubts on the notion of $\mathcal{A}-$efficiency as a useful concept to predict the behavior of the agents in real economies, as the notion of Pareto efficiency does in the Edgeworth-Coase tradition.

For the notion of Millian efficiency (and, hence, for $\mathcal{P}-$efficiency), however, a version of the First Welfare Theorem holds. To be more precise, we show in Theorem 1 that *a*) a symmetric competitive equilibrium is a –statically– Millian efficient allocation; and *b*) if long-run wages do not exceed the capitalized costs of rearing children, then a competitive equilibrium is also –dynamically– Millian efficient. Therefore, when applied to Millian efficiency, potential markets failures are of the same nature as those affecting Pareto efficient allocations in dynamic economies with exogenous fertility: although competitive equilibria are always statically Pareto efficient –i.e, it cannot be improved upon by a reallocation of the resources available for a finite number of generations–, they might be inefficient (or dynamically inefficient) –that is, they can be improved upon by a reallocation of the resources available of all generations.

Our analysis of the efficiency properties of equilibria is related to a recent paper by Schoonbroodt and Tertilt (2014), who also explore the $A-$efficiency properties of equilibria in an environment with dynastic altruism. In this paper, the authors also realize that if parents cannot obtain resources from their children, then the equilibrium outcome may be both $\mathcal{A}-$inefficient and $\mathcal{P}-$inefficient –if the utility attributed to the unborn is constant. However, they do not seem to be aware of the difficulties with the notion of $\mathcal{A}-$efficiency highlighted in this paper, and suggest that efficiency could be restored by policy measures such as fertility dependant pension schemes or a combination of public debt and fertility subsidies. The results obtained in our paper show that *a*) in many economies, restoring $\mathcal{A}-$efficiency may have undesirable consequences; *b*) symmetric equilibria may be $\mathcal{P}-$efficient for many specifications of the utility attributed to the unborn; *c*) if a symmetric equilibrium is Millian efficient, then introducing policies such as fertility dependant pension schemes cannot bring with it net welfare gains –in the sense given by the $\mathcal{A}-$dominance criterion– unless such policies discriminate among the agents; and, finally, *d*) there may exist $\mathcal{A}-$inefficient, symmetric equilibria for which the non-negativity constraints on gifts is not binding, in which case policies such as fertility dependant pension schemes might be ineffective.

The paper is organized as follows. In Section 2, we describe the setting and define the notions of $\mathcal{A}-$efficiency, $\mathcal{P}-$efficiency and Millian efficiency in the context of the setting described. In Section 3, we characterize Millian efficient allocations (in both a static and a dynamic sense). In Section 4, we highlight the above mentioned difficulties faced by the notion of $\mathcal{A}-$efficiency. In Section 5, we discuss whether or not the notion of $\mathcal{P}-$efficiency can overcome these difficulties and show that Millian efficient allocations can be regarded as $\mathcal{P}-$efficient for a wide range of specifications of the utility attributed to the unborn. In Section 6 we explore the efficient properties of decentralized equilibrium. Finally, Section 7 presents the main conclusions of the paper and discusses further research.

2 The model: assumptions and definitions

Throughout the paper, we present a slightly more general framework than GJT's extension of Barro and Becker's model. To be more precise, we consider a particular class of overlapping generations economies with infinite periods of time in which each individual lives for at most three of these periods, so that individuals living at $t = 0, 1, 2,...$ are referred to as *children, middle-aged adults* or *old adults* depending on whether t is their first, their second or their third period of life. To simplify things, all agents belong to the same dynasty, initiated by the only agent who is middle aged at $t = 0$, the *dynasty head*, hereafter represented by i^0.

As in GJT, the set of potential agents that are actually alive at any given period is endogenous and it depends on fertility plans selected by the agents. For $t = 0, 1, 2,...$, the set of possible fertility choices available to every middle-aged adult is $\mathcal{R}_+$, and the set of potential middle-aged agents at period t is $\mathcal{R}^t_+$. Each middle-aged adult *potentially* alive at $t = 1$ is identified by a positive number $i_1 \in \mathcal{R}_+$ determining the agent's position in the sibling order. For $t = 2, 3...$, each middle-aged adult potentially alive at t is identified by a vector $i^t = (i^{t-1}, i_t) \in \mathcal{R}^t_+$, where i_t specifies the agent's position in the sibling order, and $i^{t-1} = (i_1, ..., i_{t-1})$ identifies the agent's parent. Let $\mathcal{B}(\mathcal{R}^t_+)$ the class of Borelian sets in $\mathcal{R}^t_+$. For every set $B^t \in \mathcal{B}(\mathcal{R}^t_+)$ of potential middle-aged agents at t, by the measure of B^t we will refer to its Lebesgue measure $\mu_L \left\{B^t\right\} = \int_{B^t} di^t$.

A fertility plan n is a sequence of integrable functions

$$\mathsf{n} = \left\{\mathsf{n}_{t+1} : \mathcal{R}^t_+ \longrightarrow \mathcal{R}_+\right\}_{t=0,1,2,...}$$

that determines, for every t and every $i^t \in \mathcal{R}^t_+$, the number of descendants $\mathsf{n}_{t+1}(i^t)$ that agent i^t decides to have during her second period of life. Hence, for each t and every $i^t = (i^{t-1}, i_t) \in \mathcal{R}^t_+$, agent i^t is said to be alive with fertility plan n if agent i^{t-1} is also alive and $i_t \leq \mathsf{n}_t(i^{t-1})$ is satisfied. For every individual $i^{t-1} \in \mathcal{R}^{t-1}_+$ and every $\tau \geq t$, the set of descendants of i^{t-1} belonging to generation τ is denoted by $\mathcal{D}_\tau(i^{t-1})$. The set of middle-aged adults actually living at t with a fertility plan n is denoted by $\mathcal{I}_t(\mathsf{n})$ and its measure, denoted by $\mathcal{N}_t(\mathsf{n})$, is given by

$$\mathcal{N}_t(\mathsf{n}) = \mu_L\left\{\mathcal{I}_t(\mathsf{n})\right\} = \int_{\mathcal{I}_t(\mathsf{n})} di^t = \int_{\mathcal{I}_{t-1}(\mathsf{n})} \left(\int_{i_t \leq \mathsf{n}_t(i^{t-1})} di_t\right) di^{t-1} = \int_{\mathcal{I}_{t-1}(\mathsf{n})} \mathsf{n}_t(i^{t-1}) di^{t-1}.$$

With respect to the set of commodities to be allocated, we focus on the particular class of economies in which, in addition to children, there is only one homogenous good produced at every period $t \geq 1$. This consumption good is produced at each period $t = 0, 1, 2, ...$, using physical capital (K_t), i.e. the amount of the same good invested in the previous period $t-1$, and labor (L_t) provided by middle-aged adults as inputs, that is,

$$Y_t \leq F_t(K_t, L_t),$$

where Y_t is total output and $F_t : \mathcal{R}_+^2 \longrightarrow \mathcal{R}_+$ exhibits constant returns-to-scale.

Rearing children is a production activity that takes place within each household and its costs are represented by a function $b_t : \mathcal{R}_+ \longrightarrow \mathcal{R}_+$, so that a middle-aged adult who decides to rear n_{t+1} children at period t needs to spend $b_t(n_{t+1})$ units of the consumption good. Fertility and consumption plans of potential agents are represented by a fertility plan n and a sequence of integrable functions $\mathsf{c} = \left\{(\mathsf{c}_t^m, \mathsf{c}_{t+1}^o) : \mathcal{R}_+^t \longrightarrow \mathcal{R}_+^2\right\}_{t=0,1,2,...}$ that determines, for each $t = 0$, 1, 2,... and each potential agent $i^t \in \mathcal{R}_+^t$, the consumption vector $(\mathsf{c}_t^m(i^t), \mathsf{c}_{t+1}^o(i^t))$ chosen by agent i^t through her life cycle. Thus, it is assumed that children do not take consumption decisions.

The resource constraint faced by potential agents is described as follows. At time $t = 0$, the amount of resources available to finance consumption $(c_0^m(i^o))$, fertility $(n_1(i^o))$ and investment decisions $(k_1^o(i^o))$ of the dynasty head is bounded by an initial endowment $\overline{e}_0$ available for the dynasty head, that is,

$$c_0^m(i^o) + b_0(n_1(i^o)) + k_1^o(i^o) \le \overline{e}_0. \tag{1}$$

For each period $t = 0, 1, 2$, each agent who gets to be alive is exclusively endowed with 1 unit of labor time when she reaches her middle age. Then, labor is supplied inelastically, so that labor supply at any given period coincides with the measure of middle-aged agents alive at t, that is, $L_t = \mathcal{N}_t(\mathsf{n})$. Thus, at each date $t \ge 1$, the resource constraint is

$$\int_{\mathcal{I}_{t-1}(\mathsf{n})} \mathsf{c}_t^o(i^{t-1})di^{t-1} + \int_{\mathcal{I}_t(\mathsf{n})} \left(\mathsf{c}_t^m(i^t) + b_t(\mathsf{n}_{t+1}(i^t)\right) di^t + K_{t+1} \le F_t(K_t, \mathcal{N}_t(\mathsf{n})), \tag{2}$$

which by writing, for each t and each $i^t \in \mathcal{I}_t(\mathsf{n})$, $\mathsf{k}_{t+1}^o(i^t)$ for $\mathsf{k}_{t+1}^o(i^t) = \mathsf{n}_{t+1}(i^t)\frac{K_t}{\mathcal{N}_t(\mathsf{n})}$, can be equivalently written as

$$\int_{\mathcal{I}_{t-1}(\mathsf{n})} \mathsf{c}_t^o(i^{t-1})di^{t-1} + \int_{\mathcal{I}_t(\mathsf{n})} \left[\mathsf{c}_t^m(i^t) + b_t(\mathsf{n}_{t+1}(i^t)) + \mathsf{k}_{t+1}^o(i^t)\right] di^t \le \int_{\mathcal{I}_{t-1}(\mathsf{n})} F_t(\mathsf{k}_t^o(i^{t-1}), \mathsf{n}_t(i^{t-1}))di^{t-1} \tag{3}$$

In what follows, an allocation in the economy described above is a pair $\mathsf{a} = (\mathsf{x}, \mathsf{k}^o) \in \mathcal{X} \times \mathcal{K}$, where $\mathcal{X}$ is the set of sequences of the form $\mathsf{x} = \left\{\mathsf{x}_t = (\mathsf{c}_t^m, \mathsf{c}_{t+1}^o, \mathsf{n}_{t+1}) : \mathcal{R}_+^t \longrightarrow \mathcal{R}_+^3\right\}_{t=0,1,2,...}$ determining consumption and fertility choices of every potential agent and $\mathcal{K}$ is the set of sequences of the form $\mathsf{k}^o = \left\{\mathsf{k}_{t+1}^o : \mathcal{R}_+^t \longrightarrow \mathcal{R}_+^3\right\}_{t=0,1,2,...}$ determining investment decisions in every period. Without loss of generality, it is also assumed that for every t and every $i^t \in \mathcal{R}^t$ one has

$$\mathsf{x}_t(i^t) = 0 \text{ and } \mathsf{k}_{t+1}(i^t) = 0 \text{ whenever } i^t \notin \mathcal{I}_t(\mathsf{n}). \tag{4}$$

An allocation a is said to be feasible if it satisfies the initial condition in (1), the resource constraint in (3), together with condition (4). The set formed by all feasible allocations is denoted by $\mathcal{F} \subset \mathcal{X} \times \mathcal{K}$.

Throughout the paper, we assume that preferences of every potential agent of generation t on the set of allocations in which the agent is alive are represented by a utility function $\mathcal{U}_t : \mathcal{X} \times \mathcal{R}_+^t \longrightarrow \mathcal{R}$ satisfying, for every $\mathsf{x} \in \mathcal{X}$ and $i^t \in \mathcal{I}_t(\mathsf{n})$

$$\mathcal{U}_t(\mathsf{x}; i^t) = U\left(\mathsf{x}_t(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_0^{\mathsf{n}_{t+1}(i^t)} \mathcal{U}_{t+1}^D(\mathsf{x}; i^t, i_{t+1})di_{t+1}\right),$$

with $U : \mathcal{R}^3 \times \mathcal{R} \longrightarrow \mathcal{R}$, and $\mathcal{U}_{t+1}^D$ is a function representing the agent's preferences on consumption bundles made by her alive descendants. We also assume that the welfare that any agent i^t obtains from the consumption decisions of all his descendants born after t is represented by a utility function $\mathcal{U}_t^D : \mathcal{X} \times \mathcal{R}_+^t \longrightarrow \mathcal{R}$ satisfying, for every $\mathsf{x} \in \mathcal{X}$ and $i^t \in \mathcal{I}_t(\mathsf{n})$

$$\mathcal{U}_t^D(\mathsf{x}; i^t) = U^D \left(\mathsf{x}_t(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_0^{\mathsf{n}_{t+1}(i^t)} \mathcal{U}_{t+1}^D(\mathsf{x}; i^t, i_{t+1}) di_{t+1} \right),$$

where $U^D : \mathcal{R}^3 \times \mathcal{R}^* \longrightarrow \mathcal{R}^*$ is a non-decreasing function and $\mathcal{R}^* \equiv \{-\infty\} \cup \mathcal{R}$. Thus, preferences of the agents of generation t might differ from those of their parents.

To conclude the definitions, we proceed to characterize the maximal utility the dynasty head can obtain from feasible allocations. For every allocation $\mathsf{a} \in \mathcal{X} \times \mathcal{K}$ and every $t \geq 0$ and $i^t \in \mathcal{I}_t(n)$, write $e_t(i^t)$ for the amount of physical resources available for agent i^t at period t, that is,

$$e_t(i^t) := c_t^m(i^t) + b_t(n_{t+1}(i^t)) + k_{t+1}^o(i^t).$$

Consider an arbitrary i^t and e_t, and let $\mathcal{F}(e_t; i^t)$ be the set formed by all sequences $\{(\mathsf{x}_\tau, \mathsf{k}_{\tau+1}^o) : \mathcal{R}_+^t \longrightarrow \mathcal{R}_+^4\}_{\tau \geqslant t+1}$ satisfying $e_t(i^t) \leq e_t$ and, for all $\tau \geqslant t+1$, the feasibility constraint that agent i^t's descendants would face at τ if they were not allowed to obtain resources from other agents in the economy, that is

$$\int_{i^{\tau-1} \in \mathcal{D}(i^t)} \mathsf{c}_\tau^o(i^{\tau-1}) di^{\tau-1} + \int_{i^\tau \in \mathcal{D}(i^t)} \left[\mathsf{c}_\tau^m(i^\tau) + b_\tau(\mathsf{n}_{\tau+1}(i^\tau)) + \mathsf{k}_{\tau+1}^o(i^\tau) \right] di^\tau \leq \int_{i^{\tau-1} \in \mathcal{D}(i^t)} F_\tau(\mathsf{k}_\tau^o(i^{\tau-1}), \mathsf{n}_\tau(i^{\tau-1})) di^{\tau-1}.$$

For each $t \geq 1$, $e_t \geq 0$, let now $\mathcal{V}_t^D(e_t)$ be defined as the maximal utility that the dynasty head can obtain from their descendants born at t by endowing any of their immediate descendants with e_t units of resources, that is[5]

$$\mathcal{V}_t^D(e_t) = \max_{i^t \in \mathcal{R}^t} \left\{ \max_{(\mathsf{x}, \mathsf{k}^o) \in \mathcal{F}(e_t; i^t)} \mathcal{U}_t^D(\mathsf{x}; i^t) \right\};$$

which satisfies

$$\mathcal{V}_t^D(e_t) = \max_{\substack{(x_t, k_{t+1}^o) \in \mathcal{R}_+^4 \\ \mathsf{e} : \mathcal{R}_+ \longrightarrow \mathcal{R}_+}} \left\{ U^D \left(x_t, \frac{1}{n_{t+1}} \int_0^{n_{t+1}} \mathcal{V}_{t+1}^D(\mathsf{e}(i)) di \right) : c_t^m + b_t(n_{t+1}) + k_{t+1}^o \leq e_t;\ c_{t+1}^o + \int_0^{n_{t+1}} \mathsf{e}(i) di \leq F_{t+1}(k_{t+1}^o, n_{t+1}) \right\}. \tag{5}$$

With this notation, the maximum utility that the dynasty head can obtain with a feasible allocation can be written as

$$\mathcal{V}_0(\bar{e}_0) = \max_{\substack{(x_0, k_1^0) \in \mathcal{R}_+^4 \\ \mathsf{e} : \mathcal{R}_+ \longrightarrow \mathcal{R}_+}} \left\{ U \left(x_0, \frac{1}{n_1} \int_0^{n_1} \mathcal{V}_1^D(\mathsf{e}(i)) di \right) : c_0^m + b_0(n_1) + k_1^o \leq \bar{e}_0;\ c_1^o + \int_0^{n_1} \mathsf{e}(i) di \leq F_1(k_1^o, n_1) \right\}.$$

Throughout the paper, we shall impose the following additional assumptions on preferences and technologies.

[5]Since the utility received by the dynasty head from consumption of any of her descendants is the same, any choice of i^{t+1} in the optimization problem in the definition of $\mathcal{V}_t^D(e_t)$ is optimal.

*A*1 For each $t = 1, 2, ...$, the function $F_t : \mathcal{R}_+^2 \longrightarrow \mathcal{R}_+$ is linearly homogeneous, non-decreasing, concave and continuously differentiable on $\mathcal{R}_{++}^2$.

*A*2 For each $t = 1, 2, ...$, the cost function b_t is strictly increasing, convex and continuously differentiable on $\mathcal{R}_{++}$.

*A*3 For each $u^D \in \mathcal{R}$, the functions $U^D\left(\cdot, u^D\right)$ and $U\left(\cdot, u^D\right)$ are non decreasing, concave and continuously differentiable on $\mathcal{R}_{++}^3$.

*A*4 For each $u^D \in \mathcal{R}$ and any two x, $\widetilde{x} \in \mathcal{R}_+^3$, $U^D\left(x, u^D\right) \geq U^D\left(\widetilde{x}, u^D\right)$ is satisfied whenever $U\left(x, u^D\right) \geq U\left(\widetilde{x}, u^D\right)$ is satisfied.

*A*5 For any two $\left(x, u^D\right)$ and $\left(\widetilde{x}, \widetilde{u}^D\right) \in \mathcal{R}_+^3 \times \mathcal{R}$, $U\left(x, u^D\right) > U\left(\widetilde{x}, \widetilde{u}^D\right)$ is satisfied whenever $U^D\left(x, u^D\right) > U^D\left(\widetilde{x}, \widetilde{u}^D\right)$ and $u^D \geq \widetilde{u}^D$ are satisfied.

*A*6 For each t, the function $\mathcal{V}_t^D$ is well defined on $\mathcal{R}$.

As we shall see Assumptions *A*4 and *A*5 ensure that choices made by agents of generation t coincide with those that their parents would take if the latter are restricted to provide their children with a given amount of resources. But preferences of the agents of generation t and those of their parents might differ. As an example, suppose U and U^D adopt the separable form

$$U(x, u^D) = u(x) + \delta u^D$$

and

$$U^D(x, u^D) = u(x) + \beta u^D,$$

where $0 < \beta < \delta < 1$. Thus, our framework allows a wide variety of economies. We distinguish between four basic environments:

- Environments with *no altruism*, for which $U(x, u^D) = u(x)$. In this case, our framework extends the models studied by Eckstein *et al.* (1985, 1988), Michel *et al.* (2007, 2010) or Conde-Ruiz *et al.* (2010) to allow for non-symmetric allocations.
- Environments with *dynastic altruism*, for which $U(x, u^D) = U^D(x, u^D)$. In this case, our framework extends Barro and Becker's (1988, 1989) model of fertility choice, first studied by Razin and Ben Zion (1976) and also studied, among others, by Behabib and Nishimura (1989) or Schoonbroodt and Tertilt (2014).
- Environments with *finite-horizon (one-period) altruism*, for which $U(x, u^D) = u(x) + \delta u^D$ and $U^D(x, u^D) = u(x)$.
- Environments with *infinite horizon, non dynastic altruism*, an example of which arises if $U(x, u^D) = u(x) + \delta u^D$ and $U^D(x, u^D) = u(x) + \beta u^D$, with $0 < \beta < \delta < 1$.

2.1 The symmetric case

Observe that preferences and labor capacities of any two agents of the same generation are *symmetric*, that is, if any two alive agents of any generation (and all their descendants) take the same decisions, then they obtain the same utility. In view of this, it seems innocuous, both from normative and positive concerns, to restrict attention to symmetric allocations, that is, to allocations for which any two agents of the same generation choose the same consumption and fertility bundles. Moreover, since most (if not all) positive models of fertility choice exclusively focus on equilibrium notions that select symmetric allocations, restricting welfare comparisons to this particular class of allocations is still useful to evaluate the equilibrium outcomes of most models of fertility choice.

Formally, an allocation $\mathsf{a} \in \mathcal{F}$ is said to be *symmetric* if for any t and any two agents i^t, $\widetilde{i}^t \in \mathcal{I}_t(\mathsf{n})$ one has $\mathsf{x}_t(i^t) = \mathsf{x}_t(\widetilde{i}^t) = x_t$ and $\mathsf{k}_{t+1}(i^t) = \mathsf{k}_{t+1}(\widetilde{i}^t) = k_{t+1}$. Denote by $\mathcal{S} \subset \mathcal{X}^S \times \mathcal{K}^S$ the set containing all feasible symmetric allocations. A symmetric allocation is thus represented by a pair of sequences $(x, k^o) \in \mathcal{X}^S \times \mathcal{K}^S$, where $\mathcal{X}^S$ is the set of all sequences $x = \{(x_t)\}_{t=0}^{\infty}$ of non-negative vectors $x_t = (c_t^m, c_{t+1}^o, n_{t+1}) \in \mathcal{R}_+^3$ and $\mathcal{K}^S$ is the set of all sequences $k^o = \{(k_{t+1})\}_{t=0}^{\infty}$ of non-negative real numbers.

Within symmetric allocations, the measure of middle-aged agents alive at t under a symmetric fertility plan is $\mathcal{N}_t(\mathsf{n}) = \prod_{\tau=0}^{t} n_\tau = N_t$, and the resource constraint in (2) reduces to

$$N_{t-1}c_t^o + N_t\left[c_t^m + b_t(n_{t+1})\right] + K_{t+1} \leq F_t(K_t, N_t),$$

which can be equivalently written as

$$c_t^o + n_t\left[c_t^m + b_t(n_{t+1}) + k_{t+1}^o\right] \leq F_t(k_t^o, n_t), \tag{6}$$

where $k_{t+1}^o = K_{t+1}/N_t$ represents capital invested *per* old adult. A symmetric allocation (x, k^o) is feasible if it satisfies, for each $t = 0, 1, 2, ...$, the resource constraint in (6) and the initial condition $c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0$.

Note that for every t, the utility obtained by the dynasty head from consumption and fertility decisions of every two alive agents i^t and $\widetilde{i}^t$ of generation t with an allocation $\mathsf{a} \in \mathcal{S}$ satisfies $\mathcal{U}_t^D(\mathsf{x}; i^t) = \mathcal{U}_t^D(\mathsf{x}; \widetilde{i}^t) = U_t^D(x)$, where $U_t^D : \mathcal{X}^S \times \mathcal{R} \longrightarrow \mathcal{R}$ is recursively defined, for each t, by

$$U_t^D(x) = U^D\left(x_t, U_{t+1}^D(x)\right).$$

Thus, the utility obtained by an agent of generation t with a symmetric allocation is

$$U_t(x) = U(x_t, U_{t+1}^D(x)).$$

If the dynasty head is restricted to select among symmetric allocations, the maximum utility that the dynasty head can obtain with a feasible, symmetric allocation is given by

$$V_0(\overline{e}_0) = \max_{(c_0^m, k_1^o, n_1) \in \mathcal{R}_+^3, e_1 \in \mathcal{R}_+} \Big\{ U(x_0, V_1^D(e_1)) : c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0;\ c_1^o + n_1 e_1 \leq F_1(k_1^o, n_1) \Big\},$$

where, for each $t \geq 0$, $V_t^D : \mathcal{R}_+ \longrightarrow \mathcal{R}$ is defined, for every $e_t \in \mathcal{R}_+$, by

$$V_t^D(e_t) = \max_{\{(x_\tau, k_{\tau+1}^o)\}_{\tau \geq t}} \Big\{ U^D\left(x_t, U_{t+1}^D(x)\right) \ :\ c_t^m + b_t(n_{t+1}) + k_{t+1}^o \leq \widehat{e}_t;$$
$$c_\tau^o + n_\tau\left[c_\tau^m + b_\tau(n_{\tau+1}) + k_{\tau+1}^o\right] \leq F_\tau(k_\tau^o, n_\tau)\ \forall \tau \geq t+1 \Big\}.$$

A final remark is now in order. The assumptions *A*1-*A*6 do not guarantee that the dynasty head will not discriminate among her descendants and, therefore, $\mathcal{V}_0(\overline{e}_0) = V_0(\overline{e}_0)$ is satisfied. As we shall make clear throughout the paper, the set of symmetric, feasible allocations is non-convex, which in turn implies that the value functions $\mathcal{V}_t^D$ and V_t^D, that would be identical if they were both concave, might not coincide even when the function U^D is concave. In a technical Appendix, we focus on an extension of Razin and Ben Zion's model to show that, under relatively weak conditions on the elasticity of substitution of consumption goods or factors of production, the value function arising in this environment is not concave.

2.2 $\mathcal{A}-$efficiency, $\mathcal{P}-$efficiency and $\mathcal{M}-$efficiency

As noted in the Introduction, any two allocations with different fertility choices cannot be compared using the Pareto criterion, since there is no way to know whether or not an agent who lives in one allocation a but not in other allocation a' is better off in the latter than he is in the former. To avoid this problem and preserve the partial order induced by the Pareto criterion, one needs to extend it to compare also allocations of different population size. According to GJT, there are at least two possible extensions of the Pareto criterion, applicable to rank allocations with different fertility choices.

The first of these extensions, referred to as the $\mathcal{A}-$dominance criterion, ranks any two allocations by applying the Pareto criterion using information of the preference profiles of those agents who are born in the two allocations. Thus, an allocation a $\mathcal{A}-$dominates an allocation a' if for every t and every $i^t \in \mathcal{I}_t(\mathsf{n}) \cap \mathcal{I}_t(\mathsf{n}')$ one has $\mathcal{U}_t(\mathsf{x}; i^t) \geq \mathcal{U}_t(\mathsf{x}'; i^t)$, and there exists a period τ and a set $B^\tau \in \mathcal{I}_t(\mathsf{n}) \cap \mathcal{I}_t(\mathsf{n}')$ of positive measure for which $\mathcal{U}_\tau(\mathsf{x}; i^\tau) > \mathcal{U}_\tau(\mathsf{x}'; i^\tau)$ for all $i^\tau \in B^\tau$.

The second extension of the Pareto criterion proposed by GJT, the $\mathcal{P}-$dominance criterion, is constructed from a previous assumption on the utility level obtained by potential, non-born agents. This assumption, together with the utility functions that represent the preferences of the agents in those allocations in which they are alive, provides a complete description of preferences of all potential agents in the economy across all allocations, including those in which they are not alive. In order to define the notion of $\mathcal{P}-$dominance formally, let $\mathcal{U}^N$ be a sequence of functions $\mathcal{U}^N \equiv \left\{\mathcal{U}_t^N : \mathcal{X} \times \mathcal{R}^t \longrightarrow \mathcal{R}\right\}$ such that each utility function $\mathcal{U}_t^N$ in the sequence assigns, for every consumption-fertility path $\mathsf{x} \in \mathcal{X}$ and every $i^t \in \mathcal{R}^t$, a real number $\mathcal{U}_t^N(\mathsf{x}; i^t)$ that captures social judgements determining under what circumstances it is worth living.[6] Then, for any t and any potential agent of generation t, let $\mathcal{U}_t^P : \mathcal{X} \times \mathcal{N}^t \longrightarrow \mathcal{R}$ be defined, for all (x, i^t), by

$$\mathcal{U}_t^P(\mathsf{x}; i^t) = \begin{cases} \mathcal{U}_t(\mathsf{x}; i^t), & \text{if } i^t \in \mathcal{I}_t(\mathsf{n}); \\ \mathcal{U}_t^N(\mathsf{x}; i^t), & \text{otherwise.} \end{cases}$$

The notion of $\mathcal{P}-$dominance (which perhaps should be denominated $\mathcal{P}(\mathcal{U}^N)-$dominance to make explicit the specific assumption on the utility assigned to be unborn implicit in this criterion) can be defined formally as follows: an allocation a $\mathcal{P}-$dominates an allocation a' if for every t and every $i^t \in \mathcal{R}^t$ one has $\mathcal{U}_t^P(\mathsf{x}; i^t) \geq \mathcal{U}_t^P(\mathsf{x}'; i^t)$, and there exists a period τ and a set of individuals with positive measure $B^\tau \in \mathcal{R}^\tau$ for which $\mathcal{U}_\tau^P(\mathsf{x}; i^\tau) > \mathcal{U}_\tau^P(\mathsf{x}'; i^\tau)$ is satisfied for all $i^t \in B^\tau$. In most of their applications, GJT restrict the use of the term $\mathcal{P}-$dominance to the particular specification of the utilities attributed to the unborn given by

$$\mathcal{U}_t^N(\mathsf{x}; i^t) = \overline{u} \text{ for all } t = 1, 2, \ldots; \text{ and for all } (\mathsf{x}; i^t) \in \mathcal{X} \times \mathcal{R}^t.$$

These two extensions of the Pareto criterion give rise to two notions of efficiency, respectively called $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency, to evaluate allocations with different population size. A feasible allocation is $\mathcal{A}-$efficient if it is not $\mathcal{A}-$dominated by any other feasible allocation, and a feasible allocation is $\mathcal{P}-$efficient if it is not $\mathcal{P}-$dominated by any other feasible allocation. Observe that, since conditions establishing $\mathcal{P}-$dominance between any two allocations are stronger than those establishing $\mathcal{A}-$dominance, every $\mathcal{A}-$efficient allocation must be also $\mathcal{P}-$efficient. Thus, there is a wide range of allocations that are $\mathcal{P}-$efficient independently of the utility attributed

[6] Of course, any attempt for determining this utility level $\mathcal{U}_t^N(\mathsf{x}; i^t)$ constitutes a difficult task, since we cannot rely on the agents' behavior to obtain that type of information. We would like to point out, however, that being aware of these difficulties does not mean that the individuals forming a society should not decide under what circumstances it is worth living.

to the unborn. In their paper, GJT explore in detail several properties of these two notions of efficiency: first, they prove that both efficiency notions are well defined (for the notion of $\mathcal{A}-$efficiency, this property holds generically). In addition, GJT show, in the context of a setting that includes Barro and Becker's model of fertility choice as a special case, that non-cooperative equilibria selected by perfectly altruistic agents are both $\mathcal{A}-$efficient and $\mathcal{P}-$efficient.

Elsewhere (see Conde-Ruiz, Giménez and Pérez-Nievas 2004, 2010),[7] we have proposed an alternative notion of efficiency, referred to as Millian efficiency (or $\mathcal{M}$-efficiency), to evaluate symmetric allocations with different population size. This notion results from combining the $\mathcal{A}-$dominance criterion to compare allocations with the restriction of symmetry on the set of allocations that can be compared using that criterion. To be more precise,[8] a feasible symmetric allocation $\mathsf{a} \equiv \left\{ \left(x_t, k^o_{t+1} \right) \right\}_{t=0}^{\infty} \in \mathcal{S}$ is $\mathcal{M}-$efficient if there does not exist any other feasible allocation $\mathsf{a}' \in \mathcal{S}$ such that $U_t(x') \geq U_t(x)$ for all $t \geq 0$ and $U_\tau(x') > U_\tau(x)$ for some $\tau \geq 0$.

3 $\mathcal{M}-$EFFICIENCY: RELEVANT PROPERTIES

Observe that a symmetric, $\mathcal{A}-$efficient allocation must be Millian efficient. Thus, in order to explore the properties of symmetric, $\mathcal{A}-$efficient allocations, it is useful to explore first the properties of Millian efficient allocations. In this section we extend previous results (See Conde-Ruiz *et al.* , 2010) on Millian efficiency to environments with altruism.

3.1 Necessary conditions.

In this section, we present two characterizations that any Millian efficient allocation must satisfy. The first one is a condition on the utility obtained by any agent of every generation t at period t; while the second one is a condition on the dynasty head's welfare.

Recall that, for every allocation $a \in \mathcal{S}$ and every $t \geq 0$, e_t represents the amount of physical resources at period t not devoted to feed the old generation, that is,

$$e_t = c^m_t + b_t(n_{t+1}) + k^o_{t+1}.$$

With this notation, a first necessary condition for Millian efficiency can be stated as follows.

Proposition 1 *Every $\mathcal{M}-$efficient allocation $\widehat{a} \in \mathcal{S}$ satisfies, for $t \geq 0$,*

$$\begin{aligned} U_t(\widehat{x}) &= W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right) \equiv \\ &\equiv \max_{(x_t, k^o_{t+1}) \in \Re^4_+} \left\{ U\left(x_t, U^D_{t+1}(\widehat{x})\right) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} \leq \widehat{e}_t;\ F_{t+1}(k^o_{t+1}, n_{t+1}) - c^o_{t+1} \geq n_{t+1}\widehat{e}_{t+1} \right\} \end{aligned}$$

Proof. See the Appendix.

Thus, if an agent who spends an amount of resources $\widehat{e}_t$ in current consumption and investment decisions provides each of their immediate descendants with $\widehat{e}_{t+1}$ units of resources, then the pair $(\widehat{x}_t, \widehat{k}^o_{t+1})$ must solve the optimization problem in the definition of $W_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x}^t))$. For a given pair $(\widehat{x}_t, \widehat{k}^o_{t+1})$, write $\widehat{R}_{t+1}$ for $\widehat{R}_{t+1} = D_1 F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1})$ and $\widehat{w}_{t+1}$ for $\widehat{w}_{t+1} =$

[7] See also Schweizer (1997), Michel and Wigniolle (2005, 2007).

[8] In our original formulation of the notion of Millian efficiency, all those symmetric allocations for which fertility rates are zero from some period t on are also ruled out from welfare comparisons.

$D_2F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1})$. Since, utility and production functions are concave and differentiable on the interior of their domains (assumptions $A1$ and $A3$), an interior solution

$$(\widehat{x}_t, \widehat{k}_{t+1}) = \left(\widehat{x}_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})), \widehat{k}_{t+1}(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x}))\right)$$

to the optimization problem in the definition of $W_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x}))$ is characterized by the two feasibility constraints

$$\widehat{c}^m_t + b_t(\widehat{n}_{t+1}) + \widehat{k}^o_{t+1} = \widehat{e}_t, \tag{7}$$

and

$$F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}) - \widehat{c}^o_{t+1} = \widehat{n}_t \widehat{e}_{t+1}; \tag{8}$$

together with the first order conditions

$$\frac{D_1U(\widehat{x}_t, U^D_{t+1}(\widehat{x}))}{D_2U(\widehat{x}_t, U^D_{t+1}(\widehat{x}))} = \widehat{R}_{t+1}, \tag{9}$$

and

$$\left[b'_t(\widehat{n}_{t+1}) - \frac{D_3U(\widehat{x}_t, U^D_{t+1}(\widehat{x}))}{D_1U(\widehat{x}_t, U^D_{t+1}(\widehat{x}))}\right] \widehat{R}_{t+1} = \widehat{w}_{t+1} - \widehat{e}_{t+1}. \tag{10}$$

Equations (7) to (9) are almost identical to those characterizing symmetric Pareto efficient allocations in an exogenous fertility setting[9] (except for the term $b_t(\widehat{n}_{t+1})$, which in that case is assumed to be zero) and are necessary for Pareto efficiency. They simply impose feasibility and that marginal rates of substitution between current and future consumption must be equal to marginal return to investments in physical capital. The Millian notion of efficiency imposes an additional condition stating that if marginal willingness to pay for children is not equal to marginal costs of rearing children, then the marginal rate of return to investments in children must be equal to the rate of return to any other investment; that is,

$$\frac{\widehat{w}_{t+1} - \widehat{e}_{t+1}}{\left[b'_t(\widehat{n}_t) - \frac{D_3U(\widehat{x}_t, U^D_{t+1}(\widehat{x}))}{D_1U(\widehat{x}_t, U^D_{t+1}(\widehat{x}))}\right]} = \widehat{R}_{t+1}.$$

Recall, from Assumption $A4$, that $U^D(x, u^D) \geq U^D(\widetilde{x}, u^D)$ is satisfied whenever $U(x, u^D) \geq U(\widetilde{x}, u^D)$ is satisfied. Therefore, for an allocation satisfying the necessary conditions in Proposition 1 we also have, for every $t \geq 1$,

$$\begin{aligned} U^D_t(\widehat{x}) &= W^D_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right) \equiv \\ &\equiv \max_{(x_t, k^o_{t+1}) \in \Re^4_+} \left\{U^D\left(x_t, U^D_{t+1}(\widehat{x})\right) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} \leq \widehat{e}_t;\ F_{t+1}(k^o_{t+1}, n_{t+1}) - c^o_{t+1} \geq n_{t+1}\widehat{e}_{t+1}\right\}. \end{aligned}$$

This means that the utility that every alive agent (and, hence, any agent born before her) obtains from consumption decisions of all her descendants born at t is completely determined by the path $\{\widehat{e}_\tau\}_{\tau \geq t}$. Taking this into account, the following Proposition 2 restricts the set of non-negative sequences $e = \{e_t\}_{t \geq 0}$ that may correspond to an efficient allocation.

We introduce some notation first. Write e^t for the finite sequence $e^t = (e_0, e_1, e_2, ..., e_t)$; and, write e^{-t} for the infinite sequence of non-negative real numbers $e^{-t} = \{e_\tau\}_{t \geq t+1} = (e_{t+1}, e^{-(t+1)})$.

[9]See e.g., Blanchard and Fisher (1989), p.99.

With this notation, the utility that any agent obtains from consumption plans of all her descendants of generation t in a Millian efficient allocation can be written as

$$\begin{aligned} U_t^D(\widehat{x}) &= W_t^D\left(\widehat{e}_t, \widehat{e}_{t+1}, W_{t+1}^D\left(\widehat{e}_{t+1}, \widehat{e}_{t+2}, U_{t+2}^D(\widehat{x}^{-(t+1)})\right)\right) = \\ &= W_t^D\left(\widehat{e}_t, \widehat{e}_{t+1}, W_{t+1}^D\left(\widehat{e}_{t+1}, \widehat{e}_{t+2}, W_{t+2}^D\left(\widehat{e}_{t+2}, \widehat{e}_{t+3}, U_{t+3}^D(\widehat{x}^{-(t+2)})\right)\right)\right) = \\ &= ... \\ &\equiv w_t^D(\widehat{e}_t, \widehat{e^{-t}}). \end{aligned}$$

Analogously, the utility obtained by an agent of generation t can be written as

$$U_t(\widehat{x}) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, w_{t+1}^D(\widehat{e}_{t+1}, \widehat{e^{-(t+1)}})\right) \equiv w_t(\widehat{e}_t, \widehat{e^{-t}}).$$

Since, by assumption *A*6, both w_t^D and w_t must be also well-defined functions on the set formed by all infinite sequences of non-negative real numbers. Using this notation, Proposition 2 provides a second necessary condition for Millian efficiency: a Millian efficient allocation maximizes the utility of the dynasty head among all allocations that provide each generation with at least $\widehat{e}_t$ units of for which total resources available to finance their consumption, fertility and investment decisions.

Proposition 2 *Every Millian efficient allocation* $\widehat{a}$ *satisfies*

$$U_t^D(\widehat{x}) = v_t^D(\widehat{e}_t, \widehat{e^{-t}}) \equiv \max_{e^{-t} \equiv \{e_\tau\}_{\tau=t+1}^\infty} \left\{w_t^D(\widehat{e}_t, e_t^{-t}) : e^{-t} \geq \widehat{e^{-t}}\right\}, \text{ for } t \geq 1, \tag{11}$$

and

$$U_0(\widehat{x}) = v_0(\overline{e}_0, \widehat{e^{-0}}) \equiv \max_{e_1} \left\{W_0(\overline{e}_0, e_1, v_1^D(e_1, \widehat{e^{-1}})) : e_1 \geq \widehat{e}_1\right\}, \text{ for } t = 0. \tag{12}$$

Proof. See the Appendix.

3.2 Static and dynamic efficiency.

The necessary conditions in Proposition 2 are not sufficient to ensure that an allocation $\widehat{a}$ satisfying these conditions is $\mathcal{M}-$efficient. In fact, conditions (11) and (12) might not even suffice to ensure that such allocation $\widehat{a}$ satisfies a weak notion of Millian efficiency, referred to as *static* (or *short-run*) *Millian efficiency* which extends the notion of *static* –or *short-run*– *Pareto efficiency* (see Balasko and Shell 1980) to economies with endogenous fertility. Formally,

Definition 1 *A symmetric, feasible allocation* $\widehat{a} \in \mathcal{S}$ *is statically* $\mathcal{M}-$*efficient if there does not exist another symmetric, feasible allocation* $\widetilde{a} \in \mathcal{S}$ *and a finite period* $T \geq 0$ *such that*

i) $\widehat{a}_t = \widetilde{a}_t$ *for all* $t > T$;

ii) *for all* t *such that* $0 \leq t \leq T$ *one has* $U_t(\widetilde{x}) \geq U_t(\widehat{x})$*; and,*

iii) *there exists* t *such that* $0 \leq t \leq T$ *and* $U_t(\widetilde{x}) > U_t(\widehat{x})$ *is satisfied.*

Rather than providing a complete characterization of $\mathcal{M}-$efficient allocations, our main objective in this section is to provide sufficient conditions for Millian efficiency that can be applied to explore the $\mathcal{M}-$efficiency of the class of decentralized equilibria analyzed in Section 6. For this reason, we restrict ourselves to allocations satisfying a condition that, in general, is slightly stronger than that in (11), but that coincides with the latter in environments with dynastic altruism. More precisely, we focus on allocations satisfying

$$U_t(\widehat{x}) = v_t(\widehat{e}_t, \widehat{e}^{-t}) \equiv \max_{e_{t+1} \geq \widehat{e}_{t+1}} \left\{ W_t \left(\widehat{e}_t, e_{t+1}, v^D_{t+1}(e_{t+1}, \widehat{e}^{-(t+1)}) \right) \right\}, \text{ for } t \geq 0. \tag{13}$$

It is straightforward to show that, under assumption $A5$, an allocation satisfying condition (13) above satisfies also the necessary conditions (11) and (12). Also, v_t is strictly increasing in e_t and non-increasing in e^{-t}. Proposition 3 below shows that condition (13) ensures static efficiency.

Proposition 3 *An allocation satisfying $U_t(\widehat{x}) = v_t(\widehat{e}_t, \widehat{e}^{-t})$ for $t \geq 0$ is statically $\mathcal{M}-$efficient.*

Proof. See the Appendix

In the proof of Proposition 3, we show that if a statically efficient allocation $\widehat{a}$ satisfying condition (13) is not fully $\mathcal{M}-$efficient (or, as it is usually found in the literature, *dynamically* $\mathcal{M}-$efficient) there must exist an infinite sequence $\{e_t\}_{t\geq 0}$ satisfying, for some $t_1 \geq 1$ and each $t \geq t_1$

$$e_t < \widehat{e}_t$$

and

$$W_t(e_t, e_{t+1}, U^D_t(\widehat{x})) \geq W_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{x})).$$

Thus, a sufficient condition ensuring dynamic efficiency of such sequence $\{e_t\}_{t\geq 0}$ does not exist. As we have established elsewhere,[10] this sufficient condition is exactly analogous to the sufficient condition ensuring dynamic Millian efficiency of an allocation in a pure overlapping generations environment without altruism. Our results in that framework can be directly extended to the more general framework with altruism analyzed here. The main difficulty appearing in both frameworks is that, when fertility is endogenous, the instantaneous indirect utility functions $W_t\left(\cdot, U^D_t(\widehat{x})\right)$ are not, in general, quasiconcave. Due to these non-convexities, standard dynamic efficiency conditions,[11] like those requiring that the long-run interest rate $\overline{R} = D_1F(\bar{k}^o, \overline{n})$ exceeds the rate of population population growth $\overline{n}$, need not be valid to identify efficient paths.

3.3 Sufficient conditions for dynamic efficiency.

For a pair $(\widehat{e}_t, \widehat{e}_{t+1})$, define

$$\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right) = \inf_{(e_t,e_{t+1})<<(\widehat{e}_t,\widehat{e}_{t+1})} \left\{ \frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t} : W_t\left(e_t, e_{t+1}, U^D_{t+1}(\widehat{x})\right) \geq W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right) \right\}.$$

Notice that, when $W_t\left(\cdot, U^D_t(\widehat{x})\right)$ is quasiconcave, the number $\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{x})\right)$ corresponds to the slope of a indifference curve defined by $W_t\left(e_t, e_{t+1}, U^D_{t+1}(\widehat{x})\right) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right)$ evaluated at $(\widehat{e}_t, \widehat{e}_{t+1})$. That is, for quasiconcave indirect utility functions we have

$$\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})) = -\frac{D_1W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right)}{D_2W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right)} = \frac{\widehat{R}_{t+1}}{\widehat{n}_t}.$$

[10] See Lemma 1 in Conde-Ruiz *et al.* (2010, p.163).

[11] See, e.g., Phelps (1965), Koopmans (1965), and Diamond (1965).

However, it is not generally true that the indirect utility function is quasiconcave. In this case, the number $\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x}))$ corresponds to the steepest slope (on the set $(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})$) of the indifference curve defined by $W_t\left(e_t, e_{t+1}, U^D_t(\widehat{x})\right) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{x})\right)$. Therefore

$$\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})) \leq -\frac{D_1 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{x})\right)}{D_2 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{x})\right)} = \frac{\widehat{R}_{t+1}}{\widehat{n}_{t+1}}.$$

Using this notation, in Proposition 4 below we borrow directly from our previous work and provide a sufficient condition for dynamic efficiency that uses the sequence of implicit prices $\{(\widehat{R}_{t+1}, \widehat{w}_{t+1})\}_{t\geq 0}$ associated to a statically $\mathcal{M}-$efficient allocation $\widehat{a}$ satisfying (13).

Proposition 4 *Consider a statically Millian efficient allocation $\widehat{a} \in \mathcal{S}$ satisfying (13). If*

$$\lim_{T\to\infty} \inf \left(\frac{\widehat{e}_T}{\prod_{t=0}^{T} \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right)} \right) = 0 \tag{14}$$

is satisfied, then $\widehat{a}$ is also (dynamically) efficient. Furthermore, a sufficient condition ensuring (14) is that

$$\lim_{T\to\infty} \left(b'_T(0) - \frac{\widehat{w}_{T+1}}{\widehat{R}_{T+1}} \right) \frac{\widehat{n}_{t+1}}{\widehat{e}_T} > 0, \tag{15}$$

is satisfied.

Proof. See the Appendix.

4 $\mathcal{A}-$efficiency: normative concerns

In this section, we criticize the notion of $\mathcal{A}-$efficiency from normative considerations. First, we show that every symmetric, $\mathcal{A}-$efficient allocation must maximize the utility that the dynasty head can obtain with a feasible (but not necessarily symmetric) allocation. With this result in hand, we distinguish between environments with infinite horizon altruism and environments with finite horizon altruism to show that the notion of $\mathcal{A}-$efficiency faces an important difficulty: in many environments, $\mathcal{A}-$efficiency imposes that either *i*) some of the agents living in an economy obtain different consumption bundles than those obtained by their siblings, which we regard as *discrimination*; or *ii*) some of the agents devote their entire endowment (or labor capacity) to provide with resources to their parents, which we regard as *exploitation*.

The following result provides necessary conditions for $\mathcal{A}-$efficiency of a Millian efficient allocation: the only possible symmetric, $\mathcal{A}-$efficient allocation is the one maximizing the utility of the dynasty head among symmetric (and non-symmetric) allocations.

Proposition 5 *Every symmetric, $\mathcal{A}-$efficient allocation $\widehat{a}$ satisfies*

$$U^D_t(\widehat{x}) = V^D_t(\widehat{e}_t) = \mathcal{V}^D_t(\widehat{e}_t), \textit{ for } t \geq 1;$$

and

$$U_0(\widehat{x}) = V_0(\overline{e}_0) = \mathcal{V}_0(\overline{e}_0), \textit{ for } t = 0. \tag{16}$$

Proof. See the Appendix.

In the proof of Proposition 5, included in the Appendix, we proceed by showing that if a Millian efficient allocation $\widehat{a}$ does not satisfy the necessary condition (16) in the statement of Proposition 5, then it is possible to construct an alternative asymmetric allocation with more individuals that $\mathcal{M}-$dominates the allocation $\widehat{a}$. The intuition behind the Proof is simple. Suppose $U_1^D(\widehat{x}) < \mathcal{V}_1^D(\widehat{e}_1)$ is satisfied. This means that, in the Millian efficient allocation $\widehat{a}$, the dynasty head is endowing her descendants with more resources than those that would be sufficient to maximize the dynasty head's utility. Taking this into account, choose $\widetilde{e}_1 < \widehat{e}_1$ in such a way that $\mathcal{V}_1^D(\widetilde{e}_1) = U_1^D(\widehat{x})$ is satisfied and suppose that every middle-aged agent creates a new dynasty of individuals at $t = 1$. If every immediate descendant in the new dynasty is endowed with $\widetilde{e}_1$ units of resources and uses this endowment to maximize the dynasty head's utility, the latter would obtain from the new dynasty at least the same utility than the utility obtained from those descendants born in $\widehat{a}$. Moreover, since $\widetilde{e}_1 < \widehat{e}_1$, this higher utility can be obtained at a lower cost. Finally, this $\mathcal{A}-$improvement can be achieved without altering the consumption and fertility bundles of those agents already living in $\widehat{a}$. To do this, the dynasty head simply needs to change her fertility decisions and select n_1^* to maximize

$$G(n_1) \equiv \left\{ U^D \left(\bar{e}_0 - b_0(n_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, n_1) - \widehat{n}_1\bar{e}_0 - (n_1 - \widehat{n}_1)\,\bar{e}_0, n_1, \frac{\widehat{n}_1}{n_1} U_1^D(\widehat{x}) + \left(1 - \frac{\widehat{n}_1}{n_1}\right) \mathcal{V}_1^D(\widetilde{e}_1) \right) \right\},$$

among all numbers $n_1 \geq \widehat{n}_1$. That is, the dynasty head simply has to select her consumption and fertility plans to maximize her utility, given that she has to provide with $\widehat{e}_1$ units of resources to every agent already born under $\widehat{a}$ and with $\widetilde{e}_1$ units of resources to all agents in the new dynasty. It is straightforward to show that if the allocation $\widehat{a}$ is Millian efficient, then the function G is strictly increasing on the interval $[0, \widehat{n}_1]$. Moreover, its left hand side derivative at $\widehat{n}_1$ satisfies $\frac{d^- G(\widehat{n}_1)}{dn_1} = 0$, while the right hand side derivative satisfies $\frac{d^+ G(\widehat{n}_1)}{dn_1} > 0$. Since n_1^* is chosen to maximize $G(n_1)$, this implies that the agents born at $t = 1$ will obtain at least the same utility than the utility they obtain with $\widehat{a}$. Also, the fact that $n_1^* > \widehat{n}_1$ is satisfied implies that each new dynasty of individuals has positive measure. Thus, a symmetric, $\mathcal{A}-$efficient allocation must satisfy the necessary condition (16) in Proposition 5. Taking this into account, a similar argument shows that condition (16) is also necessary for symmetric $\mathcal{A}-$efficiency.

A straightforward implication of Proposition 5 is that the set of symmetric, $\mathcal{A}-$efficient allocations is significantly smaller than the set of Millian efficient allocations. This contrasts with the set of symmetric, Pareto efficient allocations in settings with exogenous populations, which includes all those symmetric allocations that are not Pareto dominated by any other symmetric allocation. Observe that achieving welfare gains –in the sense given by the $\mathcal{A}-$dominance criterion– from a Millian efficient allocation $\widehat{a}$ makes it necessary to discriminate between those agents who are alive under $\widehat{a}$ and those who are not.

The following two results explore the sequences of Proposition 5 in environments with finite-horizon and infinite-horizon altruism, respectively.

Proposition 6 *In an environment with finite horizon altruism, the set of interior, symmetric, $\mathcal{A}-$efficient allocations is empty. Moreover, in every symmetric, $\mathcal{A}-$efficient allocation, the economy collapses in finite time.*

Proof. To prove Proposition 6, simply observe that on an environment satisfying, for some β and δ such that $0 = \beta \leq \delta < 1$,

$$U^D(x, u^D) = u(x) + \beta u^D = u(x)$$

and

$$U(x, u^D) = u(x) + \delta u^D,$$

Proposition 5 implies that a symmetric, $\mathcal{A}-$efficient allocation $\widehat{a}$ must satisfy

$$U_1^D(\widehat{x}) = \mathcal{V}_1^D(\widehat{e}_1) = V_1^D(\widehat{e}_1) = W_1^D(\widehat{e}_1, 0, U^D(0)) = w_1^D(\widehat{e}_1, 0),$$

which, in turn, yields

$$x_t = 0 \text{ for each } t \geq 2,$$

which establishes that the symmetric allocation maximizing the utility of the dynasty head must be non-interior and, therefore, completes the proof of Proposition 6.□

Thus, in this type of environments, achieving $\mathcal{A}-$efficiency without discrimination requires that all agents of a generation are exploited, in the sense that they devote their entire endowment –or labor capacity– to provide with resources to their parents. This type of exploitation implies that achieving $\mathcal{A}-$efficiency leads the economy to a collapse in finite time. Furthermore, although we restrict attention to symmetric allocations that are also $\mathcal{A}-$efficient, the construction in the proof of Proposition 5 suggests that, even if we allow for discrimination, achieving $\mathcal{A}-$efficiency in environments with finite horizon altruism imposes some form of exploitation. The following result shows that the set of symmetric, $\mathcal{A}-$efficient allocations might be empty even if the agents' preferences exhibit infinite-horizon altruism.

Proposition 7 *Let $\widehat{a}$ be the allocation maximizing the utility of the dynasty head among symmetric allocations and suppose that there exists a period T and an amount of physical resources $\widetilde{e}_{T+1} \neq \widehat{e}_{T+1}$ for which*

$$V_{T+1}^D(\widetilde{e}_{T+1}) - V_{T+1}^D(\widehat{e}_{T+1}) > \frac{dV_{T+1}^D(\widehat{e}_{T+1})}{de}(\widetilde{e}_{T+1} - \widehat{e}_{T+1}) \tag{17}$$

is satisfied. In this case, the set of symmetric, $\mathcal{A}-$efficient allocations is indeed empty.

Proof. To prove Proposition 7, it is straightforward to show from (5) and Proposition 1 that the unrestricted value function $\mathcal{V}_t^D$ satisfies, for every $t > 0$ and every $e_t \geq 0$, the Bellman equation

$$\mathcal{V}_t^D(e_t) = \max_{h:\mathcal{R}_+ \longrightarrow [0,1] \in \triangle\mathcal{R}_+} W_t^D\left(e_t, \int_{\mathcal{R}_+} e\, dh(e), \int_{\mathcal{R}_+} \mathcal{V}_{t+1}^D(e)\, dh(e)\right),$$

where $\triangle\mathcal{R}_+$ represents the set of distribution functions with support in $\mathcal{R}_+$. Using this representation of the value function, a straightforward application of Jensen's inequality makes clear that a sufficient condition ensuring that the functions $\mathcal{V}_t^D$ and V_t^D coincide is that each function $\mathcal{V}_\tau^D$ in the sequence $\{\mathcal{V}_\tau^D\}_{\tau \geq t+1}$ is concave. Conversely, the functions $\mathcal{V}_t^D$ and V_t^D may not coincide if V_τ^D is not concave. To see this, assume, without loss of generality, that each function in the sequence $\{V_t^D\}_{t\geq 0}$ is differentiable and let $\{\widehat{e}_\tau\}_{\tau \geq t+1}$ be a path corresponding to a solution to the optimization problem in the definition of $V_t^D(\widehat{e}_t)$. Then suppose that there exists a period T and an amount of physical resources $\widetilde{e}_{T+1} \neq \widehat{e}_{T+1}$ for which (17) is satisfied. Finally, for a number $\pi \in (0, 1)$, write $U_{\widehat{e}_t}^D(\pi)$ for

$$U_{\widehat{e}_t}^D(\pi) = W_t^D\left(\widehat{e}_t, \pi\widetilde{e}_{t+1} + (1-\pi)\widehat{e}_{t+1}, \pi V_{t+1}^D(\widetilde{e}_{t+1}) + (1-\pi)V_{t+1}^D(\widehat{e}_{t+1})\right).$$

Observe that $U_{\widehat{e}_t}(0) = V_t^D(\widehat{e}_t)$. Also,

$$\frac{dU^D_{\widehat{e}_t}(\pi)}{d\pi} = D_2W_t^D\left(\widehat{e}_t, \widehat{e}_{t+1}, V^D_{t+1}(\widehat{e}_t)\right)\left[\widetilde{e}_{\tau+1} - \widehat{e}_{\tau+1}\right] + D_3W_t^D\left(\widehat{e}_t, \widehat{e}_{t+1}, V^D_{t+1}(\widehat{e}_t)\right)\left[V^D_{t+1}(\widetilde{e}_{t+1}) - V^D_{t+1}(\widehat{e}_{t+1})\right] > 0,$$

which yields, for some π^* such that $0 < \pi^* \leq 1$,

$$\mathcal{V}_t^D(\widehat{e}_t) \geq U^D_{\widehat{e}_t}(\pi^*) > V_t^D(\widehat{e}_t).$$

In view of Proposition 5, this establishes that the set of symmetric, $\mathcal{A}-$efficient allocations is empty and, hence, completes the proof of Proposition 7.□.

Under what conditions are the value functions concave? Even if we restrict ourselves to environments with dynastic altruism, growth models of fertility choice depart from standard growth models in that *i*) the set of feasible allocations is non-convex, and in that *ii*) the factor by which the dynasty head discounts the utility of her descendants is endogenous, which, in turn, implies that the dynastic utility functions might be non-concave. Although some authors –see e.g., Benhabib and Mishimura (1989), or Nishimura and Raut (1999)– have noticed that these non-convexities may generate strange dynamics, only a few papers have explicitly studied whether or not value functions arising in standard models are concave. Some exceptions are Álvarez (1999), Schoonbroodt and Tertilt (2014) and Qi and Kadaya (2010). The first two papers focus on environments in which preferences are homothetic, and provide conditions -on the rate of growth of feasible paths- ensuring that show that value functions are concave. The latter provide conditions ensuring the concavity of value functions in a general setting, but it does it at a cost: fertility choices must be bounded from below.

In a Technical Appendix[12], we study whether or not value functions are concave in a setting in which (*a*) the agents live for two periods, and (*b*) preferences are strictly concave, homothetic and additively separable. As shown in the Appendix, value functions arising in this context are concave when the interest rate over the rate of population growth is a decreasing function of the capital stock (in *per worker* terms). However, as Michel and Wigniolle (2007, 2010) or our own work (2010) shows, seemingly well-behaved preferences may give rise to economies in which the the interest rate over the rate of population growth is a non-monotonic function of the capital stock. In the Appendix, we show that in such economies, value functions may be non-concave. Of course, problems originated by non convexities may be exacerbated in non-standard models that take into account other important features of fertility choice, such as non-divisible choice, externalities, etc.

5 $\mathcal{P}-$EFFICIENCY: AVOIDING SOME REPUGNANT CONSEQUENCES OF $\mathcal{A}-$EFFICIENCY

In the previous section, we have shown that for many allocations, it is possible to find an asymmetric allocation with more individuals in which some people who were already alive in the former allocation are better off in the new allocation, and no alive agent are worse-off. Since the notion of $\mathcal{P}-$dominance limits the way in which new people can be brought up to the economy, one possible way to avoid the negative results in the previous Section is by using the notion of $\mathcal{P}$-efficiency instead of $\mathcal{A}-$efficiency. For example, consider a sequence of utility functions satisfying Assumption 4.*a*) in GJT; that is, satisfying

$$\mathcal{U}_t^N(\mathsf{x}; i^t) = \overline{u} \text{ for all } t \geq 1. \tag{18}$$

[12]The technical Appendix is available from the authors upon request.

With this specification of the utility attributed to the unborn, a Millian efficient allocation $\widehat{a}$ for which $U_t(\widehat{x}) \leq \overline{u}$ for all $t \geq 1$ cannot be $\mathcal{P}-$dominated by an allocation with higher population size, because achieving a $\mathcal{P}-$improvement by increasing the population requires to provide all newcomers with more resources than the resources they have in $\widehat{a}$. Analogously –at least, in environments with altruism–, there is no way to achieve a $\mathcal{P}-$improvement by decreasing the population size because all the remaining living agents (the dynasty head among them) would get lower utility than that they obtain with $\widehat{a}$. Hence, the allocation $\widehat{a}$ is $\mathcal{P}$-efficient.

When the utility function attributed to the unborn adopts the particular specification in (18), the way by which the notion of $\mathcal{P}-$efficiency mitigates the problems raised in the previous Section is not entirely satisfactory. First, because if ir is the case that $U_t(\widehat{x}) < \overline{u}$ for all $t \geqslant 1$, we find somewhat strange accepting as optimal an allocation in which all the living agents are strictly worse off than those who are not born (no matter how the individuals in a society come to that judgement). Second, and most important, because determining whether or not an allocation is optimal (i.e. $\mathcal{P}-$efficient) becomes heavily dependent on that judgement.[13]

However, other specifications of $\mathcal{U}^N$ may reduce this dependency, as long as the utility attributed to the unborn depends on the utility achieved by those alive at any allocation. There are several possibilities. For instance, the principles

(i) "No one should be born if being born makes her worse-off than any of her living siblings born before her;"

(ii) "Whenever (i) cannot be applied, no one should be born if being born makes her worse-off than any other member of her generation;" and,

(iii) "Whenever (i) or (ii) cannot be applied, no one should be born if being born makes her obtain lower utility than a utility threshold $\overline{u}$,"

could be captured by selecting a function defined, for every $(\mathsf{x}, i^t) \in \mathcal{X} \times \mathcal{R}^{t+1}$, by

$$\mathcal{U}_t^N(\mathsf{x}; i^{t-1}, i_t) = \begin{cases} \max\left\{\mathcal{U}_t(\mathsf{x}; i^{t-1}, i_\tau) : i_\tau < i_t; i_\tau \leq \mathsf{n}_t(i^{t-1})\right\}, & \text{if } \mathsf{n}_t(i^{t-1}) > 0, \\ \max\left\{\mathcal{U}_t(\mathsf{x}; i^t) : i^t \in \mathcal{I}_t(\mathsf{n})\right\} & \text{if } \mathsf{n}_t(i^{t-1}) = 0 \text{ and } \mathcal{N}_t(\mathsf{n}) > 0, \\ \overline{u}, & \text{otherwise.} \end{cases}.$$

(i) and (ii) –and keep principle (iii)–, by

(i') "No one should be born if she enjoys a lower welfare level than the average welfare level enjoyed by her living siblings born before her;" and

(ii') "Whenever (i') cannot be applied, no one should be born if being born makes her worse off than a representative member of her generation,"

which can be represented by selecting a utility function for the unborn that adopts the *Average Utilitarianism* form,

$$\mathcal{U}_t^N(\mathsf{x}; i^{t-1}, i_t) = \begin{cases} \frac{1}{\mathsf{n}_t(i^{t-1})}\left(\int_{i_\tau \leq \mathsf{n}_t(i^{t-1})} \mathcal{U}_t(\mathsf{x}; i^{t-1}, i_\tau) di_\tau\right), & \text{if } \mathsf{n}_t(i^{t-1}) > 0, \\ \frac{1}{N_t(\mathsf{n})}\left(\int_{i^t \in \mathcal{I}_t(\mathsf{n})} \mathcal{U}_t(\mathsf{x}; i^t) di^t\right), & \text{if } \mathsf{n}_t(i^{t-1}) = 0 \text{ and } \mathcal{N}_t(\mathsf{n}) > 0, \\ \overline{u}, & \text{otherwise.} \end{cases}.$$

[13] In the social choice literature, a similar solution has been proposed, in response to the Repugnant Conclusion, by the use of *critical level population principles* (Blackorby, Bossert and Donaldson 2002, 2005) on social welfare functions.

Finally, a third possibility arises by replacing (i) and (ii) –and keeping principle (iii)–, by the principles

(i'') "No one should be born if being born makes her worse-off than her living siblings born before her who obtain the lowest utility;"

(ii'') "Whenever (i'') cannot be applied, no one should be born if being born makes her worse-off than any other living member of her generation,"

which can be represented by selecting a utility function for the unborn that adopts the *Rawlsian* form,

$$\mathcal{U}_t^N(\mathsf{x};i^{t-1},i_t)=\begin{cases}\min\left\{\mathcal{U}_t(\mathsf{x};i^{t-1},i_\tau):i_\tau<i_t;i_\tau\leq\mathsf{n}_t(i^{t-1})\right\}, & \text{if } \mathsf{n}_t(i^{t-1})>0,\\ \min\left\{\mathcal{U}_t(\mathsf{x};i^t)\ :\ i^t\in\mathcal{I}_t(\mathsf{n}),\right\}, & \text{if } \mathsf{n}_t(i^{t-1})=0 \text{ and } \mathcal{N}_t(\mathsf{n})>0,\\ \overline{u}, & \text{otherwise.}\end{cases}.$$

In all these examples, the function determining the utility obtained by the unborn (as a function of decisions made by living agents) in a given allocation has the following common property:

Property 1. *For all $t\geq1$, every $i^t=(i^{t-1},i_t)\in\mathcal{R}_+^t$ and any two allocations a and a', if $i_\tau>\mathsf{n}'_t(i^{t-1})>\mathsf{n}_t(i^{t-1})$ and $\mathcal{U}_t(\mathsf{x};i^{t-1},i_t)\geq\mathcal{U}_t(\mathsf{x}',i^{t-1},i_t)$ for every $i_t\in[0,i_\tau]$, then*

$$\mathcal{U}_t^N(\mathsf{x};i^{t-1},i_\tau)\geq\mathcal{U}_t^N(\mathsf{x}';i^{t-1},i_\tau).$$

Property 2. *For every $t\geq1$, every $i^t\in\mathcal{R}_+^t$ and every interior, symmetric allocation a such that $\mathsf{x}_t(i^t)=x_t>0$ and $\mathsf{n}_{t+1}(i^t)=n_{t+1}>0$ one has*

$$\mathcal{U}_t^N(\mathsf{x};i^t)=\mathcal{U}_t(\mathsf{x};i^t)\equiv U_t(x).$$

The first of these properties simply requires that the utility attributed to a particular agent when she does not get to be born depends (monotonically) on the utility profiles of those that are born before her. Without this property (for example, if the utility attributed to an agent i^t if unborn in a given allocation depends also on the utility obtained by those siblings that would also be unborn if i^t was unborn) the binary relation induced by the $\mathcal{P}-$dominance criterion might be intransitive, even when we restrict welfare comparisons to symmetric allocations. The second property simply establishes that the utility attributed to a particular agent if unborn is a symmetric function of the utility profiles of her living siblings.

Proposition 8 below provides conditions under which, for any specification of the utilities attributed to the unborn that satisfies the above two properties, every interior, Millian efficient allocation is also $\mathcal{P}-$efficient. A new definition is needed first. For an arbitrary $t\geq0$, let $\mathbf{v}_t^D(\widehat{e}_t,\widehat{e}^{-t})$, be defined, for every sequence $(\widehat{e}_t,\widehat{e}^{-t})$, by

$$\mathbf{v}_t^D(\widehat{e}_t,\widehat{e}^{-t})\ =\ \max_{i^t\in\mathcal{R}^t}\left\{\max_{(\mathsf{x},\mathsf{k}^o)\in\mathcal{F}(\widehat{e}_t;i^t)}\left\{\mathcal{U}_t^D(\mathsf{x};i^t):\mathsf{e}_\tau(i^\tau)\geq\widehat{e}_\tau\text{ for all }\tau\geq t+1\right\}\right\}.$$

That is, $\mathbf{v}_t^D(\widehat{e}_t,\widehat{e}^{-t})$ is the maximum utility that agents born before t can obtain from consumption decisions of their descendants, provided each of these descendants is endowed with at least $\widehat{e}_\tau$ units of resources.

Proposition 8 *Let $\mathcal{U}^N$ be a sequence of functions satisfying Properties* 1 *and* 2*, and let $\widehat{a}$ be a Millian efficient allocation. If the function $\mathbf{v}_t^D$ is concave on the set of sequences $\{e_t : e_t \geq \widehat{e}_t\}_{t\geq 1}$, then $\widehat{\mathsf{a}}$ is $\mathcal{P}-$ efficient (with respect to $\mathcal{U}^N$).*

Proof. See the Appendix.

That is, Millian efficient allocations may be also $\mathcal{P}-$efficient. Furthermore, they are $\mathcal{P}-$efficient as long as each function $\mathcal{U}_t^N$ belongs to the class of functions satisfying Properties 1 and 2. Thus, just as an $\mathcal{A}-$efficient allocation can be described as a $\mathcal{P}-$efficient allocation for which $\mathcal{P}-$efficiency holds irrespectively of the utility attributed to the unborn, Millian efficient allocations can be described as $\mathcal{P}-$efficient allocations for which $\mathcal{P}-$efficiency holds for a wide range of specifications of the utility attributed to the unborn.

6 Equilibrium behavior: a failure in the First Welfare Theorem?

After exploring the properties of the three notions of efficiency proposed in the literature, we conclude the paper by studying the efficiency properties of a decentralized mechanism in which the agents, endowed with well-defined property rights over the commodities available in the economy, are free to trade these rights (or transfer them) to pursue their own interests. With this objective, we explore the efficiency properties of a notion of decentralized equilibrium that, as in GJT, results from the combination of the notion of competitive equilibrium and the notion of subgame perfect equilibrium of a voluntary transfer game played within families. Differently from GJT, we impose that gifts cannot be negative and parents cannot condition their gifts and bequests on their children's behavior.

In our notion of decentralized equilibrium, there are two markets operating at each date $t \geq 0$: a financial market, that allows agents to lend (or borrow) an arbitrary amount $s_t^o = k_{t+1}^o$ of the homogeneous good in period t, and obtain (or pay back) a return equal to $R_{t+1}k_{t+1}^o$ units of the same good in period $t+1$; and, a spot job market, in which labor is exchanged against the homogeneous good at a price w_t. Since the agents may exhibit descendant altruism, each type i^t of an agent of generation t might be willing to transfer, at period $t+1$, an amount $\mathsf{g}_{t+1}(i^t, i_{t+1}) \geq 0$ of the *numeraire* to each of her immediate descendants when they reach their middle age, which we may refer to as a bequest or a gift depending on whether or not the agents live for one or two periods.

If the agents hold correct expectations both on future prices (represented by a sequence $p^{-t} \equiv \{w_\tau, R_\tau\}_{\tau=t+1,\ldots}$) and on their descendants' future consumption decisions, represented by a sequence of functions $\mathsf{x}^{-t} = \left\{\mathsf{x}_t : \mathcal{R}^{t+1} \longrightarrow \mathcal{R}_+\right\}_{t\geq t+1}$, then an agent of generation t who receives a gift from their parents $\mathsf{g}_t(i^t)$ and transfers resources to her descendants according to a gift scheme $\mathsf{g}_{t+1} : \mathcal{R}_+^{t+1} \longrightarrow \mathcal{R}_+$ will choose her consumption-fertility bundle $\mathsf{x}_t^*(i^t)$ and her savings $\mathsf{k}_{t+1}^*(i^t)$ to solve

$$\max_{(x_t,k_{t+1}^o)\in\Re_+^4} \left\{U\left(x_t, \frac{1}{n_{t+1}}\int_0^{n_{t+1}} \mathcal{U}_{t+1}^D(\mathsf{x}; i^t, i_{t+1})di_{t+1}\right) \ : \ c_t^m + b_t(n_{t+1}) + k_{t+1}^o = w_t + \mathsf{g}_t(i^t); \right. \tag{19}$$

$$\left. c_{t+1}^o = R_{t+1}k_{t+1}^o - \int_0^{n_{t+1}} \mathsf{g}_{t+1}(i^t, i_{t+1})di_{t+1}\right\}.$$

By Assumption $A4$, any solution $\left(\mathsf{x}_t^*(i^t), \mathsf{k}_{t+1}^*(i^t)\right)$ to the optimization problem above is also a solution of the optimization problem in which the objective function in (19) is replaced by $U^D(x_t, \frac{1}{n_{t+1}}\int_0^{n_{t+1}} \mathcal{U}_{t+1}^D(\mathsf{x}; i^t, i_{t+1})di_{t+1})$. By proceeding recursively, it follows that the sequence of prices $p = \{w_t, R_t\}_{t=1,...}$, together with the gift $\mathsf{g}_t(i^t)$ that the agent i^t receives from her parents and the sequence of gift schemes $\mathsf{g}^{-t} = \{\mathsf{g}_\tau : \mathcal{R}^\tau \longrightarrow \mathcal{R}\}_{\tau \geq t+1}$ determining the gift received by all her descendants, fully determines both the consumption-fertility bundle $\mathsf{x}_t^*(i^t) = x_{p,t}\left(\mathsf{g}_t(i^t), \mathsf{g}^{-t}\right)$ as well as the amount saved as capital $\mathsf{k}_{t+1}^*(i^t) = k_{p,t+1}\left(\mathsf{g}_t(i^t), \mathsf{g}^{-t}\right)$ selected by i^t. Thus, the pair $\left(\mathsf{g}_t(i^t), \mathsf{g}^{-t}\right)$ also determines the (indirect) utility payoffs $\Pi_{p,t}(\mathsf{g}_t(i^t), \mathsf{g}^{-t})$ that agents born at t obtain from a sequence of gift schemes $\mathsf{g} = \left\{\mathsf{g}_{\tau+1} : \mathcal{R}^{t+1} \longrightarrow \mathcal{R}_+\right\}_{\tau \geq 0}$.

To obtain an explicit expression of the sequence of payoff functions, write $n_{p,t+1}(\mathsf{g}_t(i^t), \mathsf{g}^{-t})$ for the fertility plan that solves the optimization problem in (19) and write $\mu_{p,t+1}(\cdot/\mathsf{g}_t(i^t), \mathsf{g}^{-t})$ for the distribution function defined, for each $g \in \mathcal{R}_+$, by

$$\mu_{p,t+1}(g/\mathsf{g}_t(i^t), \mathsf{g}^{-t}) = \frac{1}{n_{p,t+1}(\mathsf{g}_t(i^t), \mathsf{g}^{-t})} \int_{i_{t+1}:\mathsf{g}_{t+1}(i^t, i_{t+1}) \leq g}^{n_{p,t+1}(\mathsf{g}_t(i^t), \mathsf{g}^{-t})} di_{t+1}.$$

Also, let $W_{p,t} : \mathcal{R}_+^3 \longrightarrow \mathcal{R}$ and $W_{p,t}^D : \mathcal{R}_+^3 \longrightarrow \mathcal{R}$ be respectively defined, for each $(e_t, e_{t+1}, U_{t+1}^D)$, by

$$\begin{aligned} W_{p,t}(e_t, e_{t+1}, U_{t+1}^D) = \max_{(x_t, k_{t+1}^o) \in \Re_+^4} \Big\{ U(x_t, U_{t+1}^D) \quad : \quad & c_t^m + b_t(n_{t+1}) + k_{t+1}^o = e_t; \\ & c_{t+1}^o = R_{t+1} k_{t+1}^o - n_{t+1}[w_{t+1} - e_{t+1}] \Big\}, \end{aligned}$$

and

$$\begin{aligned} W_{p,t}^D(e_t, e_{t+1}, U_{t+1}^D) = \max_{(x_t, k_{t+1}^o) \in \Re_+^4} \Big\{ U^D(x_t, U_{t+1}^D) \quad : \quad & c_t^m + b_t(n_{t+1}) + k_{t+1}^o \leq e_t; \\ & c_{t+1}^o = R_{t+1} k_{t+1}^o - n_{t+1}[w_{t+1} - e_{t+1}] \Big\}. \end{aligned}$$

Using this notation, the (indirect) utility payoffs $\Pi_{p,t}\left(\mathsf{g}_t(i^t), \mathsf{g}^{-t}\right)$ that agents born at t obtain from a sequence of gift schemes g can be written as

$$\Pi_{p,t}\left(\mathsf{g}_t(i^t), \mathsf{g}^{-t}\right) = W_{p,t}\left(w_t + \mathsf{g}_t(i^t), w_{t+1} + \int_{\mathcal{R}} g d\mu_{p,t+1}(g/\mathsf{g}_t(i^t), \mathsf{g}^{-t}), \int_{\mathcal{R}} \Pi_{p,t+1}^D(g, \mathsf{g}^{-(t+1)}) d\mu_{p,t+1}(g/\mathsf{g}_t(i^t), \mathsf{g}^{-t})\right). \tag{20}$$

where $\Pi_{p,t+1}^D(g_{t+1}, \mathsf{g}^{-(t+1)})$ is recursively defined by

$$\Pi_{p,t}^D(\mathsf{g}_t(i^t), \mathsf{g}^{-t}) = W_{p,t}^D\left(w_t + \mathsf{g}_t(i^t), w_{t+1} + \int_{\mathcal{R}} g d\mu_{p,t+1}(g/\mathsf{g}_t(i^t), \mathsf{g}^{-t}), \int_{\mathcal{R}} \Pi_{p,t+1}^D(g, \mathsf{g}^{-(t+1)}) d\mu_{p,t+1}(g/\mathsf{g}_t(i^t), \mathsf{g}^{-t})\right). \tag{21}$$

In our notion of equilibrium, gift schemes chosen by the agents constitute a subgame perfect equilibrium of the the game of voluntary transfers Γ_p in which, for each $t = 0, 1, ...$, each living agent i^t of generation t decides a gift scheme $\mathsf{g}_{t+1}:\mathcal{R}_+^{t+1} \longrightarrow \mathcal{R}_+$, which yields her a payoff determined by the function $\Pi_{p,t}(\mathsf{g}_t(i^t), \mathsf{g}^{-t})$. To be more precise, we impose that the agents play Symmetric Markov Strategies [hereafter, SMS], such that, for each t an any arbitrary i^t, agent i^t's equilibrium gift scheme satisfies, for each i_{t+1},

$$\mathsf{g}_{t+1}(i^t, i_{t+1}) = G_{t+1}(\mathsf{g}_t(i^t)).$$

If all agents play the strategy determined by $G = \{G_{t+1} : \mathcal{R}_+ \to \mathcal{R}_+\}_{t \geq 0}$, the utility obtained by an agent of generation t –from the gift schemes chosen by her descendants– by giving g_{t+1} units of resources to each of her immediate descendants depends only on g_{t+1} and can be recursively written as

$$\Pi^D_{p,G,t+1}(g_{t+1}) = W^D_{p,t+1}(w_{t+1} + g_{t+1}, w_{t+2} + G_{t+2}(g_{t+1}), \Pi^D_{p,G,t+2}(G_{t+2}(g_{t+1})).$$

Thus, the sequence of strategies G corresponds to a Subgame Perfect Equlibria of the game played within families if for each $t > 0$ and each $g_t > 0$, $G_{t+1}(g_t)$ solves

$$\Pi_{p,G,t}(g_t) \equiv \max\left\{W_{p,t}(w_t + g_t, w_{t+1} + g_{t+1}, \Pi^D_{p,G,t+1}(g_{t+1})) : g_{t+1} \geq 0\right\}, \text{ for } t > 0; \tag{22}$$

while the strategy selected by the dynasty head solves

$$\Pi_{p,G,0}(\overline{e}_0) \equiv \max\left\{W_{p,t}(\overline{e}_0, w_1 + g_1, \Pi^D_{p,G,1}(g_1)) : g_1 \geq 0\right\}. \tag{23}$$

Given a sequence of prices p and a sequence $\{e_t\}_{t\geq 0}$, let $\left\{w_{p,t}(e_t, e^{-t})\right\}_{t\geq 1}$ be a sequence of functions recursively defined, for each t, by

$$\begin{aligned} w^D_{p,t}(e_t, e^{-t}) &= W^D_{p,t}\left(e_t, e_{t+1}, w^D_{p,t+1}(e_{t+1}, e^{-(t+1)})\right) = \\ &= W^D_{p,t}\left(e_t, e_{t+1}, W^D_{p,t}\left(e_{t+1}, e_{t+2}, w^p_{t+2}(e_{t+2}, e^{-(t+2)})\right)\right). \end{aligned}$$

Also, for a given sequence of prices p and a sequence $\{\widehat{g}_t\}_{t\geq 1}$ corresponding to a SMS, let $\{\widehat{e}_t\}_{t\geq 1}$ be the sequence defined, for each $t \geq 1$, by $\widehat{e}_t = w_t + \widehat{g}_t$, and write $V^D_{p,G,t}(\widehat{e}_t)$ and $V_{p,G,t}(\widehat{e}_t)$, respectively, for $V^D_{p,G,t}(\widehat{e}_t) = \Pi^D_{p,G,t}(\widehat{e}_t - w_t)$ and $V_{p,G,t}(\widehat{e}_t) = \Pi_{p,G,t}(\widehat{e}_t - w_t)$. Note that, if $\{\widehat{e}_t\}_{t\geq 1}$ is the sequence of income flows generated by a sequence of Symmetric, Markov Strategies $\{\widehat{g}_t\}_{t\geq 1}$, then we have

$$V^D_{p,G,t+1}(\widehat{e}_{t+1}) = \Pi^D_{p,G,t+1}(\widehat{e}_{t+1} - w_{t+1}) = W^D_{p,t}\left(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{p,t+1}(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)})\right) = w^D_{p,t+1}(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)}).$$

With this notation, conditions (22) and (23) can be written, in terms of the sequence $\{\widehat{e}_t\}_{t\geq 1}$ corresponding to an equilibrium as

$$V_{p,G,t}(\widehat{e}_t) = W_{p,t}\left(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{p,t+1}(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)})\right) = \max\left\{W_{p,t}\left(\widehat{e}_t, e_{t+1}, w^D_{p,t+1}(e_{t+1}, \widehat{e}^{-(t+1)})\right) : e_{t+1} \geq w_{t+1}\right\} \tag{24}$$

and

$$V_{p,G,0}(\widehat{e}_t) = W_{p,G,t}(\overline{e}_0, \widehat{e}_1, w^D_{p,1}(\widehat{e}_1, \widehat{e}^{-1})) = \max\left\{W_{p,1}(\overline{e}_0, e_1, w^D_{p,1}(e_1, \widehat{e}^{-1})) : e_1 \geq w_1\right\} \tag{25}$$

We should point out that conditions (22) and (23) are not sufficient to guarantee that a sequence of Markov strategies G satisfying these conditions constitutes a Subgame perfect equilibrium of Γ_p. A sufficient condition ensuring that a sequence of strategies G satisfying (22) and (23) is a subgame perfect equilibrium of the dynastic game is that each function $\Pi^D_{p,G,t+1}$ is concave on $\mathcal{R}_+$. Observe that concavity of $\Pi^D_{p,G,t+1}$ holds if $V^D_{p,G,t+1}$ is concave on the interval $[w_{t+1}, \infty)$.

Moreover, even if we restrict ourselves to payoffs for which each function $V^D_{p,G,t+1}$ is concave on $[w_{t+1}, \infty)$, there might be many equilibria of the dynastic game. To see this, suppose the functions U and U^D (and, hence, the functions $V_{p,t}$ and $V^D_{p,t}$) coincide. In this case, any sequence $\{e_t\}_{t\geq 1}$ of functions $\{V_{p,G,t+1} : \mathcal{R}_+ \longrightarrow \mathcal{R}\}$ satisfying

$$V_{p,G,t}(\widehat{e}_t) = \max\left\{W_{p,t}(\widehat{e}_t, e_{t+1}, V_{p,G,t+1}(e_{t+1})) : e_{t+1} \geq w_{t+1}\right\}, \text{ for } t > 0, \tag{26}$$

may correspond to the payoffs obtained by the agents in an equilibrium with SMS.

In general, there exist many solutions to the sequence of functional equations in (26), each of them corresponding to the utility payoffs of a Subgame Perfect Equilibrium (with symmetric, Markov strategies) of the dynastic game. To select among these, in the notion of decentralized equilibrium that we propose here we shall assume that, at each point in time t, the agents choose the SMS that maximizes their utilities among all possible sequences of SMS's that correspond to Subgame Perfect Equilibria of the game Γ_p.[14] In this equilibrium, the utility $V_{p,t}^D(\widehat{e}_t)$ that all agents born before period t obtain from consumption decisions of their descendants satisfies, for $t \geqslant 1$

$$V_{p,t}^D(\widehat{e}_t) = \max_{\{e_\tau\}_{\tau \geqslant t}} \Big\{ W_t^D(e_t, e_{t+1}, w_{p,t+1}^D(e_{t+1}, e^{-(t+1)})) : e_t \leq \widehat{e}_t; e^{-t} \geq w^{-t};$$
$$W_\tau(e_\tau, e_{\tau+1}, w_{p,\tau+1}^D(e_{\tau+1}, e^{-(\tau+1)})) = \max\Big\{W_\tau(e_\tau, \widetilde{e}_{\tau+1}, w_{p,\tau+1}^D(\widetilde{e}_{\tau+1}, e^{-(\tau+1)})) : \widetilde{e}_{\tau+1} \geq w_{\tau+1}\Big\}$$
$$\text{for all } \tau \geq t.\Big\}$$

Therefore,

$$V_{p,t}^D(\widehat{e}_t) = \max_{e_{t+1} \geq 0} \Big\{ W_t^D\left(\widehat{e}_t, e_{t+1}, V_{p,t+1}^D(e_{t+1})\right) : e_{t+1} \geq w_{t+1};$$
$$W_t(\widehat{e}_t, e_{t+1}, V_{p,t+1}^D(e_{\tau+1})) = \max_{\widetilde{e}_{t+1} \geq 0} \left\{W_t\left(\widehat{e}_t, \widetilde{e}_{t+1}, V_{p,t+1}^D(\widetilde{e}_{t+1})\right) : \widetilde{e}_{t+1} \geq w_{t+1}\right\} \Big\}.$$

Observe that, if $U \equiv U^D$, the equality constraint in the definition of $V_{p,t}^D(\widehat{e}_t)$ becomes redundant and, therefore,

$$V_{p,t}^D(\widehat{e}_t) = V_{p,t}(\widehat{e}_t) = \max\Big\{W_t^D(\widehat{e}_t, e_{t+1}, w_{p,t+1}^D(e_{t+1}, e^{-(t+1)})) : e^{-t} \geq w^{-t}\Big\},$$

which, in turn, implies that the sequence $\{\widehat{e}_t\}_{t \geq 0}$ corresponding to the equilibrium satisfies

$$V_{p,t}(\overline{e}_0) = V_{p,t}^D(\overline{e}_0) = \max\Big\{W_0^D(\overline{e}_0, e_1, w_{p,1}^D(e_1, e^{-1})) : e^{-1} \geq w^{-1}\Big\}.$$

Equilibrium behavior, however, differs from dynastic maximization if $U \neq U^D$. Suppose, for example, that $U^D(x, U^D) = u(x) + \beta U^D$ and $U(x, U^D) = u(x) + \delta U^D$ are satisfied, with $0 \leq \beta < \delta < 1$. In this environment, we have

$$\begin{aligned} W_{p,t}(e_t, e_{t+1}, U_t^D(x_{t+1})) &= W_t(e_t, e_{t+1}) + \delta U_t^D(x_{t+1}) = \\ &= W_{p,t}^D(e_t, e_{t+1}, U_t^D(x_{t+1})) + (\delta - \beta)\, U_t^D(x_{t+1}). \end{aligned}$$

Taking this into account, it is straightforward to show that the sequence $\{\widehat{e}_t\}_{t \geq 0}$ corresponding to the equilibrium solves the optimization problem

$$\max\left\{\sum_{\tau=0}^{\infty} \delta^t W_t(e_t, e_{t+1}) : e_0 = \overline{e}_0 : e_t \geq w_t \ \forall t \geqslant 1.\right\}.$$

Thus, our notion of symmetric competitive equilibrium generalizes the notion of equilibrium implicitly used by Barro and Becker (1988) to a setting in which altruism might be of the non-dynastic type. In our model, however, the nonnegativity constraint on intrafamiliar transfers might be binding in equilibrium, even when parents exhibit infinite horizon, dynastic altruism.

To summarize, a *symmetric decentralized equilibrium* can be defined formally as a feasible, symmetric allocation $\widehat{a}$ and a sequence of prices $p \equiv \{w_t, R_t\}_{t=1,2,\ldots}$ such that

[14] Using arguments to those used by Fudemberg and Levine (1983), it can be shown that our notion of equilibrium corresponds to that Subame Perfect Equilibrium of Γ_p, that is also the limit of the finite horizon truncations of the game in which transfers are constrained to be zero from some period T on.

i) for each $t \geq 0$, the pair $(\widehat{x}_t, \widehat{k}^o_{t+1})$ maximizes utility, that is, is selected to satisfy

$$U\left(\widehat{x}_t, U^D_{t+1}(\widehat{x}_{t+1})\right) = W_{p,t}\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x}_{t+1})\right),$$

and the pair $(\widehat{k}^o_{t+1}, \widehat{n}_{t+1})$ maximizes profits, that is, satisfies

$$D_1 F_{t+1}(\widehat{k}_{t+1}, \widehat{n}_{t+1}) = R_{t+1} \text{ and } D_2 F_{t+1}(\widehat{k}_{t+1}, \widehat{n}_{t+1}) = w_{t+1};$$

ii) the sequence $\{\widehat{e}_t\}$ of resources available to each generation corresponds to the Subgame Perfect Equilibrium of the dynastic game in which the agents select the strategy that maximizes their utilities among all possible equilibria of the dynastic game in which the agents play Symmetric Markov Strategies, that is,

$$V_{p,t}(\widehat{e}_t) = \max\left\{W_{p,t}(\widehat{e}_t, e_{t+1}, V^D_{p,t+1}(e_{t+1})) : e_{t+1} \geq w_{t+1}\right\} \text{ for all } t \geq 0, \tag{27}$$

where, for each $t \geq 0$, $V^D_{p,t+1} : \mathcal{R}_+ \longrightarrow \mathcal{R}$ is defined, for each e_{t+1}, as in (22);

iii) for each $t \geq 0$, the function $V^D_{p,t}$ is concave on $[w_{t+1}, \infty)$.

Thus, even if we allow for non-symmetric strategies, the interaction of markets and families in the framework analyzed in the paper delivers, under relatively weak concavity conditions on each value function $V^D_{p,t}$, a symmetric, interior allocation. In view of our results in previous sections, this implies that the First Welfare Theorem does not hold (at least, when applied to $\mathcal{A}-$efficiency) in environments with non-dynastic altruism. With dynastic altruism, the First Welfare Theorem might still hold if, at equilibrium prices, the non-negativity constraint on gifts is not binding and $V_{p,t}(\widehat{e}_t) = V_t(\widehat{e}_t) = \mathcal{V}_t(\widehat{e}_t)$ is satisfied. Otherwise, competitive equilibria are $\mathcal{A}-$inefficient.

The possibility that competitive equilibria arising with voluntary (and non-negative) transfers are $\mathcal{A}-$inefficient has also been noticed by Schoonbroodt and Tertilt (2014), who view this possibility as a potential market failure –that arises because parents have no claim on their children's labor income– and propose to correct this market failure with (fertility dependant) pension schemes. Our previous results, however, suggest that, if altruism is of the non-dynastic type, then the only way of achieving $\mathcal{A}-$efficiency is by allowing parents to appropriate their children's entire labor income (in case parents are not altruistic towards their children) or by allowing parents to condition the gifts or bequests transferred to their children to the gifts or bequests that the latter decide to transfer to their own children (in case altruism is non-dynastic). Observe also that, when agents are not altruistic or their altruism is of the finite-horizon type, the only way of restoring efficiency through the introduction of fertility dependant pension schemes –which delivers a symmetric allocation– would lead the economy to a collapse in finite time.

Under the assumption that the unborn agents always obtain lower utility that any born agent, $\mathcal{A}-$inefficient decentralized equilibria are $\mathcal{P}-$inefficient as well. However, Theorem 1 below shows that decentralized equilibria may be Millian efficient and, hence, $\mathcal{P}$-efficient.

Theorem 1 *Let $\widehat{a}$ be an allocation corresponding to a symmetric competitive equilibrium.*
i) $\widehat{a}$ is statically Millian efficient;
ii) If $\widehat{a}$ satisfies condition (14) or condition (15), then it is also dynamically efficient.

Thus, in absence of dynamic efficiency problems –that may arise also in economies with exogenous population– the First Welfare Theorem holds for Millian efficiency (or $\mathcal{P}-$efficiency), rather than for $\mathcal{A}-$efficiency.

7 Conclusions

In the paper, we have explored the properties of the notions of $\mathcal{A}-$efficiency and $P-$efficiency, proposed by Golosov, Jones and Tertilt (*Econometrica*, 2007), as well as the notion of Millian efficiency (Conde-Ruiz, Giménez and Pérez-Nievas, 2010) to evaluate allocations in a general overlapping generations setting with endogenous fertility and descendant altruism. The setting includes, as particular cases, environments with infinite horizon, dynastic altruism *à la* Barro and Becker (1988), as well as environments with finite horizon or other forms of non-dynastic altruism. By allowing for general specifications, we have shown that if we evaluate social welfare without making any judgement on whether or not it is worth living –that is, if we use the notion of $\mathcal{A}-$efficiency– an important difficulty arises: in many environments, the set of symmetric, interior, $\mathcal{A}-$efficient allocations is empty. This means that achieving $\mathcal{A}-$efficiency imposes that either *i*) some of the agents living in an economy obtain different consumption bundles than those obtained by their siblings, even when these siblings have the same tastes and capacities; or *ii*) some of the agents devote their entire endowment (or labor capacity) to provide with resources to their parents. Moreover, the class of environments for which either *i*), which can be regarded as discrimination, or *ii*), which can be regarded as exploitation, are necessary conditions for $\mathcal{A}-$efficiency includes all environments with finite horizon altruism, but also economies with infinite horizon, dynastic altruism.

Then, we have shown that these difficulties can be overcome if we incorporate principles determining under what circumstances it is worth living –that is, if we use the notion of $\mathcal{P}-$efficiency. For a wide range of functions determining the welfare attributed to the unborn, every Millian efficient allocation, that is, every symmetric allocation that is not $\mathcal{A}-$dominated by any other symmetric allocation, is $\mathcal{P}$-efficient. Finally, we have provided a version of the First Welfare Theorem by showing that *a*) every symmetric competitive equilibrium is a –statically– Millian efficient allocation; and, that *b*) if long run wages do not exceed the capitalized costs of rearing children, then competitive equilibria are also –dynamically– Millian efficient.

Thus, the notion of Millian efficiency (or $\mathcal{P}-$efficiency) seems more appropriate than that of $\mathcal{A}-$efficient to evaluate allocations specially in settings in which altruism is not of the dynastic type; with the notion of Millian efficiency an important qualitative conclusion of GJT prevails: in absence of non convexities, externalities, missing markets, dynamic efficiency problems, etc., the fact that fertility is endogenous does not mean that markets fail to deliver efficient allocations.

There are several directions that might be worth exploring. A first direction would be to explore whether or not the results arising here can be extended to other models of fertility choice, for example those in which parents exhibit some form of altruism towards their ancestors, as in Caldwell (1982). It does not seem that this type of altruism would eliminate all problems raised in this paper because it does not rule out the possibility that parents "exploit" their descendants by appropriating their entire endowment. A second direction would be to extend the results to environments in which the agents are heterogeneous. Here, we should point out that the symmetry restriction underlying the Millian notion of efficiency imposes that every two agents with the same characteristics should be treated equally, but it does not preclude that agents with different characteristics must be treated equally. Thus, in models in which agents are heterogeneous in their characteristics (preferences, endowments, preferences and endowments of their ancestors and finally, the agents' order of birth with respect to their siblings), the Millian notion of efficiency may be still applicable if we regard the symmetry restriction as imposing that any two agents of the same generation with the same preferences and endowments –and for whom the preferences and endowments of all their ancestors are also equal– must be treated symmetrically.

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Appendix: Proofs

Proof of Proposition 1. To prove Proposition 1, suppose that $\widehat{a}$ is an $\mathcal{M}-$efficient allocation, and suppose that there exists a period $\tau \geq 0$ for which the pair $(\widehat{x}_\tau, \widehat{k}^o_{\tau+1})$ corresponding to the allocation $\widehat{a}$ is not a solution to the optimization problem in the definition of $W_\tau(\widehat{e}_\tau, \widehat{e}_{\tau+1}, U^D_{\tau+1}(\widehat{x}))$. Select now a solution $(\widetilde{x}_\tau, \widetilde{k}^o_{\tau+1}) \in \mathcal{R}^4_+$ to such optimization problem and let $\widetilde{a}$ be the allocation obtained from $\widehat{a}$ by replacing the term $(\widehat{x}_\tau, \widehat{k}^o_{\tau+1})$ by such solution. This allocation is feasible because $(\widetilde{x}_\tau, \widetilde{k}^o_{\tau+1})$ must satisfy $\widetilde{c}^m_\tau + b_t(\widetilde{n}_{\tau+1}) + \widetilde{k}^o_{\tau+1} \leq \widehat{e}_\tau$ and $F_{t+1}(\widetilde{k}^o_{t+1}, \widetilde{n}_{t+1}) - \widetilde{c}^o_{t+1} \geq \widetilde{n}_{t+1}\widehat{e}_{t+1}$. Also, observe that, by assumption $A4$, the fact that $U_\tau(\widetilde{x}) = U(\widetilde{x}, U^D_\tau(\widetilde{x})) > U(\widehat{x}, U^D_\tau(\widehat{x})) = U_\tau(\widehat{x})$ is satisfied, implies $U^D_\tau(\widetilde{x}) = U^D(\widetilde{x}, U^D_\tau(\widetilde{x})) > U^D(\widehat{x}, U^D_\tau(\widehat{x})) = U^D_\tau(\widehat{x})$ is satisfied. Therefore $U^D_t(\widetilde{x}) \geq U^D_t(\widehat{x})$ for all $t \geq \tau$ and, since both U and U^D are monotonic, $U^D_t(\widetilde{x}) > U^D_\tau(\widehat{x})$ and $U_t(\widetilde{x}) \geq U_\tau(\widehat{x})$ for all $t < \tau$. That is, if the term $(\widehat{x}_\tau, \widehat{k}^o_{\tau+1})$ is not a solution to the optimization problem in the definition of $W_\tau(\widehat{e}_\tau, \widehat{e}_{\tau+1}, U^D_{\tau+1}(\widehat{x}))$, then $\widehat{a}$ is $\mathcal{M}-$dominated by an alternative allocation $\widetilde{a}$, a contradiction that establishes Proposition 1.□

Proof of Proposition 2. To prove Proposition 2, let $\widehat{a}$ be a Millian efficient allocation. Observe that, by Proposition 1 and Assumption $A4$ we have $U^D_t(\widehat{x}) = w^D_t(\widehat{e}_t, \widehat{e}^{-t})$ for each $t \geq 0$. To prove that $U^D_t(\widehat{x}) = v^D_t(\widehat{e}_t, \widehat{e}^{-t})$ must be satisfied for each $t \geq 1$, suppose that there exists a period $\tau \geq 0$ for which the sequence $\{\widehat{e}_t\}^\infty_{t=\tau}$ associated to the allocation $\widehat{a}$ is not a solution to the optimization problem in the definition of $W^D_\tau(\widehat{e}_\tau, \widehat{e}_{\tau+1}, U^D_{\tau+1}(\widehat{x}))$. Select now a sequence $\widetilde{e}$ such that $w^D_\tau(\widehat{e}_\tau, \widetilde{e}^{-\tau}) > w^D_\tau(\widehat{e}_\tau, \widehat{e}^{-\tau})$ is satisfied for $\tau \geq 0$, and let now $\widetilde{a}$ be the symmetric allocation for which $U^D_t(\widetilde{x}) = w^D_t(\widetilde{e}_t, \widetilde{e}^{-t})$ is satisfied for $t \geq 0$. Note that

$$U^D_t(\widetilde{x}) = W^D_t\left(\widetilde{e}_t, \widetilde{e}_{t+1}, U^D_{t+1}(\widetilde{x})\right) \geq W^D_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right) = U^D_t(\widehat{x})$$

must be satisfied for each $t \geq 1$. Moreover, the above inequality must be satisfied as a strict inequality for $t = \tau$. Finally, assumption $A5$ and the fact that $U^D_t(\widetilde{x}) \geq U^D_t(\widehat{x})$ is satisfied implies that

$$U_t(\widetilde{x}) = W_t\left(\widetilde{e}_t, \widetilde{e}_{t+1}, U^D_{t+1}(\widetilde{x})\right) \geq W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right) = U_t(\widehat{x})$$

is also satisfied for each $t \geq 1$, and with strict inequality for $t = \tau$. This implies that the allocation $\widehat{a}$ is not $\mathcal{M}-$efficient, a contradiction that establishes that $U^D_t(\widehat{x}) = v^D_t(\widehat{e}_t, \widehat{e}^{-t})$ must be satisfied. Taking (11) into account, (12) follows straightforwardly, which completes the proof of Proposition 2.□

Proof of Proposition 3. Let $\widehat{a} \in \mathcal{S}$ be an allocation satisfying condition (13). To prove that $\widehat{a}$ is statically $\mathcal{M}-$efficient, we proceed by showing that if there exists an allocation $\widetilde{a}$ that $\mathcal{M}-$dominates the allocation $\widehat{a}$, then there must exist an infinite subsequence $\mathcal{T} = \{t_1, t_2, t_3, ...\}$ such that $\widetilde{e}_t < \widehat{e}_t$, and $U_t(\widetilde{x}) \geq U_t(\widehat{x})$ for each $t \in \mathcal{T}$. To prove this statement, observe that the fact that $U_0(\widehat{x}) = v_0(\overline{e}_0, \widehat{e}^{-0})$ and the fact that v_0 is non-increasing in e^{-0} is satisfied imply that $\widetilde{e}_{t_1} < \widehat{e}_{t_1}$ must be satisfied for some period $t_1 \geq 0$. Since v_{t_1} is strictly increasing in e_{t_1} and non-increasing in e^{-t_1}, the fact that $\widetilde{e}^{-t_1} \geq \widehat{e}^{-t_1}$ is satisfied yields

$$U_{t_1}(\widetilde{x}) < v_{t_1}(\widehat{e}_{t_1}, \widehat{e}^{-t_1}) = U_{t_1}(\widehat{x}),$$

which contradicts the assumption imposing that $\widetilde{a}$ $\mathcal{M}-$dominates the allocation $\widehat{a}$. Therefore, $U_{t_1}(\widetilde{x}) \geq U_{t_1}(\widehat{x})$ can be satisfied if there exists t_2 for which $\widetilde{e}_{t_2} < \widehat{e}_{t_2}$ and $W^D_{t_2}(\widetilde{e}_{t_2}, \widetilde{e}_{t_2+1}, U^D_{t_2+1}(\widetilde{x})) > U_{t_2}(\widehat{x})$ is satisfied. By applying the argument recursively, the existence of the subsequence $\mathcal{T}$ is established. Also, since the allocation $\widehat{a}$ can only be dominated by a reallocation of resources involving a infinite sequence of periods of time, the allocation $\widehat{a}$ must be statically $\mathcal{M}-$efficient, which establishes Proposition 3.□

Proof of Proposition 4. In environments with no altruism, that is, for which preferences are represented by a utility function satisfying $\mathcal{U}(x, u^D) = u(x)$ and the sequence of indirect utility functions adopts the form $W_t(e_t, e_{t+1}, U^D_t(x)) = W_t(e_t, e_{t+1})$ and $\pi_t(e_t, e_{t+1}, U^D_t(x)) = \pi_t(e_t, e_{t+1})$, Proposition 4 corresponds to Proposition 3 in Conde-Ruiz *et al.* (2010). The proof proceeds by steps.

Step 1). In a first step, we show that, if there exists an allocation $\widetilde{a}$ that $\mathcal{M}-$dominates the allocation $\widehat{a}$, then there must exist an allocation a that also dominates $\widehat{a}$ and satisfies, for $t \geq 1$

$$U_t(x) = U_t(\widehat{x}), \tag{28}$$

$$U_t^D(x) \leq U_t^D(\widehat{x}), \tag{29}$$

and

$$e_t < \widehat{e}_t. \tag{30}$$

To prove this statement, assume, without loss of generality, that $\widetilde{a}$ satisfies the necessary conditions for Millian efficiency in Propositions 1 and 2. Taking this into account, an allocation a satisfying the required properties can be constructed from the allocation $\widetilde{a}$ as follows. Pick up any period $\tau \geqslant 1$ for which $U_\tau(\widetilde{x}) = W_\tau(\widetilde{e}_\tau, \widetilde{e}_{\tau+1}, U_{\tau+1}^D(\widetilde{x})) > U_\tau(\widehat{x})$ is satisfied and select $e_\tau^1 < \widetilde{e}_\tau$ in such a way that $W_\tau(e_\tau^1, \widetilde{e}_{\tau+1}, U_{\tau+1}^D(\widetilde{x})) = U_\tau(\widehat{x})$ is satisfied. Notice that, by Assumption $A5$ one must have $W_\tau^D(e_\tau^1, \widetilde{e}_{\tau+1}, U_{\tau+1}^D(\widetilde{x})) \leq U_\tau^D(\widehat{x})$. Then let x^1 be the allocation obtained from $\widetilde{a}$ by replacing the term $(\widetilde{x}_\tau, \widetilde{k}_\tau^o)$ by the solution (x_τ^1, k_τ^{o1}) to the optimization problem in the definition of $W_\tau(e_\tau^1, \widetilde{e}_{\tau+1}, U_{\tau+1}^D(\widetilde{x}))$. Note that, since $\widetilde{a}$ satisfies the necessary conditions in Propositions 1 and 2 and $w_{\tau-1}^D$ is non-increasing in e_t, we have $U_{\tau-1}^D(x^1) = W_{\tau-1}^D(\widetilde{e}_{t-1}, e_\tau^1, W_\tau^D(e_\tau^1, \widetilde{e}_{\tau+1}, U_{\tau+1}^D(\widetilde{x}))) \geq U_\tau^D(\widetilde{x})$. Thus, $U_{\tau-1}^D(x^1) \geq U_{\tau-1}^D(\widehat{x})$ and, hence, $U_{\tau-1}(x^1) \geq U_{\tau-1}(\widehat{x})$ must be satisfied, that is, the allocation x^1 dominates the allocation $\widetilde{a}$. Proceeding iteratively, it is straightforward to construct an allocation $\overline{a}$ satisfying the required properties for each $t : 1 \leq t \leq \tau$, which taking into account that τ has been selected arbitrarily among those periods for which $U_t(\overline{x}) > U_t(\widehat{x})$ is satisfied, establishes Step 1).

Step 2). In a second Step, we show that for any allocation a (28) to (30) for $t \geq 1$ we have,

$$\frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t} = \pi_t(e_t, e_{t+1}, U_t^D(\widehat{x})), \text{ for t} \geq 1.$$

To prove Step 2), observe that $\pi_t(e_t, e_{t+1}, U_t^D(\widehat{x}))$ has been defined as the highest value of the ratio $\frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t}$ among those pairs $(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})$ in the indifference curve defined by

$$W_t\left(e_t, e_{t+1}, U_{t+1}^D(\widehat{x})\right) = U_t(\widehat{x}). \tag{31}$$

Observe that if $U_t^D(x) < U_t^D(\widehat{x})$, the indifference curve defined by $W_t\left(e_t, e_{t+1}, U_{t+1}^D(x)\right) = U_t(\widehat{x})$ lies below the indifference curve in (31). Therefore, for any allocation a satisfying (28)-(30) and any $t \geq 1$, we have

$$\frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t} \geq \pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})), \text{ for t} \geq 1.$$

This establishes Step 2).

Step 3). We now show that if $\widehat{a}$ is dynamically efficient satisfying (13) and (14) in the statement of Proposition 4 then an allocation a satisfying conditions (28)-(30) cannot exist. To show that this yields a contradiction, observe that $e_\tau < \widehat{e}_\tau$ must be satisfied, and therefore there exists ϵ such that $\epsilon = \widehat{e}_1 - e_1 > 0$. Since $\widehat{a}$ satisfies condition (14), there must exist an infinite subsequence $\mathcal{T} \equiv \{t_0, t_1, ...\}$ of periods of time and a sufficiently large $T^* \geq 1$ such that, for each $T \in \mathcal{T}$ such that $T > T^*$ one has

$$\left(\frac{\widehat{e}_T}{\prod_{t=1}^T \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)}\right) < \epsilon = \widehat{e}_1 - e_1. \tag{32}$$

Use Step 2) and the definition of $\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)$ to obtain, for any T, the chain of inequalities

$$\begin{aligned} 0 &< (\widehat{e}_1 - \widetilde{e}_1) = \epsilon \leq \frac{(\widehat{e}_2 - \widetilde{e}_2)}{\pi_1\left(\widehat{e}_1, \widehat{e}_2, U_2^D(\widehat{x})\right)} \leq \frac{\widehat{e}_3 - \widetilde{e}_3}{\pi_1\left(\widehat{e}_1, \widehat{e}_2\right)\pi_2\left(\widehat{e}_2, \widehat{e}_3, U_3^D(\widehat{x})\right)} \leq \\ &\leq \ ... \leq \\ &\leq \frac{\widehat{e}_T - \widetilde{e}_T}{\prod_{t=1}^T \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)} < \frac{\widehat{e}_T}{\prod_{t=1}^T \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)}, \end{aligned}$$

which, for $T \in \mathcal{T}$ contradicts condition (32) and, therefore, establishes that $\widehat{a}$ is Millian efficient.

Step 4). The remaining of the proof follows an exactly analogous argument to that in Proposition 5 in Conde-Ruiz *et al* (2010).□

Proof of Proposition 5. To prove Proposition 5, let $\widehat{a}$ be a symmetric, $\mathcal{A}-$efficient allocation. To show that $U_t^D(\widehat{x}) = \mathcal{V}_t^D(\widehat{e}_t)$ must be satisfied for each $t \geq 1$, observe that if $U_t^D(\widehat{x}) < \mathcal{V}_t^D(\widehat{e}_t)$ for some period $t > 1$, then $U_1^D(\widehat{x}) < \mathcal{V}_1^D(\widehat{e}_1)$ is also satisfied. To show that $U_1^D(\widehat{x}) = \mathcal{V}_1^D(\widehat{e}_1)$ must be satisfied, suppose it is not, that is, suppose $U_1^D(\widehat{x}) < \mathcal{V}_1^D(\widehat{e}_1)$ is satisfied. Then select a pair $(\widetilde{n}_0, \widetilde{e}_1) \geq 0$ such that $\widetilde{e}_1 < \widehat{e}_1$ and $U_1^D(\widetilde{x}) = \mathcal{V}_1^D(\widetilde{e}_1)$ are satisfied and consider the (possibly, asymmetric) allocation $\widetilde{\mathsf{a}}$ constructed from the allocation $\widehat{a}$ and the pair $(\widetilde{n}_1, \widetilde{e}_1)$ as follows:

i) for $t = 0$,

$$\begin{aligned}
\mathsf{k}_1^o(i^0) &= \widehat{k}_1^o, \\
\mathsf{c}_0^m(i^0) &= \overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, \\
\mathsf{c}_1^o(i^0) &= F_1(\widehat{k}_1^o, \widetilde{n}_1) - \widehat{n}_1\widehat{e}_1 - (\widetilde{n}_1 - \widehat{n}_1)\,\widetilde{e}_1;
\end{aligned}$$

ii) for $t = 1$ and $i^1 \in \mathcal{I}_1(\widetilde{\mathsf{n}})$,

$$\left(\mathsf{x}_1(i^1), \mathsf{k}_2^o(i^1), \mathsf{e}_1(i^1)\right) = \begin{cases} (\widehat{x}_1, \widehat{k}_2^o, \widehat{e}_1), & \text{if } i \in \mathcal{I}_1(\widehat{\mathsf{n}}), \\ (\widetilde{x}_1, \widetilde{k}_2^o, \widetilde{e}_1), & \text{if } i \in \mathcal{I}_t(\widetilde{\mathsf{n}}) \backslash \mathcal{I}_t(\widehat{\mathsf{n}}), \end{cases}$$

where $(\widetilde{x}_1, \widetilde{k}_2^o)$ corresponds to the solution $\left(\widetilde{x}_1, \widetilde{k}_2^o, \widetilde{\mathsf{e}}_2(i^1, \cdot)\right)$ to

$$\mathcal{V}_1^D(\widetilde{e}_1) = \max_{(x_1, k_2^o, \mathsf{e}_2(i^1,\cdot))} \Bigg\{ U^D\left(x_1, \frac{1}{n_2}\int_0^{n_2} \mathcal{V}_2^D(\mathsf{e}_2(i^1, i_2))di_2\right) : \\ c_1^m + b_1(n_2) + k_2^o \leq \widetilde{e}_1;\ c_2^o + \int_0^{n_2} \mathsf{e}_2(i^1, i_2)di_2 \leq F_2(k_2^o, n_2)\Bigg\};$$

iii) for $t > 1$ and each $i^t \in \mathcal{I}_t(\widetilde{\mathsf{n}})$,

$$\left(\mathsf{x}_t(i^t), \mathsf{k}_{t+1}^o(i^t), \mathsf{e}_t(i^t)\right) = \begin{cases} (\widehat{x}_t, \widehat{k}_{t+1}^o, \widehat{e}_t), & \text{if } i \in \mathcal{I}_t(\widehat{\mathsf{n}}), \\ (\widetilde{x}_t(i), \widetilde{k}_{t+1}^o(i), \widetilde{e}_t(i)), & \text{if } i \in \mathcal{I}_t(\widetilde{\mathsf{n}}) \backslash \mathcal{I}_t(\widehat{\mathsf{n}}), \end{cases}$$

where $(\widetilde{x}_t(i), \widetilde{k}_{t+1}^o(i))$ corresponds to the solution $\left(\widetilde{x}_t, \widetilde{k}_{t+1}^o, \widetilde{\mathsf{e}}_{t+1}(i^t, \cdot)\right)$ to

$$\mathcal{V}_t^D(\widetilde{e}_t(i^t)) = \max_{(x_t, k_{t+1}^o, \mathsf{e}_{t+1}(i^t,\cdot))} \Bigg\{ U^D\left(x_t, \frac{1}{n_{t+1}}\int_0^{n_{t+1}} \mathcal{V}_{t+1}^D(\mathsf{e}_{t+1}(i^t, i_{t+1}))di_{t+1}\right) : \\ c_t^m + b_t(n_{t+1}) + k_{t+1}^o \leq \widetilde{e}_t(i^t); \\ c_{t+1}^o + \int_0^{n_{t+1}} \mathsf{e}_{t+1}(i^t, i_{t+1})di_{t+1} \leq F_{t+1}(k_{t+1}^o, n_{t+1})\Bigg\}.$$

Note that, for $\widetilde{n}_1 < \widehat{n}_1$, the allocation $\widetilde{\mathsf{a}}$ is a symmetric allocation with fewer individuals than those living in $\widehat{a}$; while for $\widetilde{n}_1 > \widehat{n}_1$, $\widetilde{\mathsf{a}}$ is an asymmetric allocation that splits the population into (at least) two groups at period $t = 1$. To be more precise, when $\widetilde{n}_1 > \widehat{n}_1$ is satisfied, those living under $\widehat{a}$ obtain the same consumption-fertility bundle that the one they obtain in $\widehat{a}$, while those who were not living under $\widehat{a}$ receive an endowment $\widetilde{\mathsf{e}}_t(i^t)$ and take the consumption and fertility plans that maximize the utility of the dynasty head. It is straightforward to see that the allocation $\widetilde{\mathsf{a}}$ is feasible.

Write now $G(\widetilde{n}_1)$ for the (indirect) utility obtained by the dynasty head with the allocation $\widetilde{\mathsf{a}}$ for different selections of $\widetilde{n}_1$. Since $U_1^D(\widehat{x}_1) = V_1^D(\widehat{e}_1)$ we have

$$G(\widetilde{n}_1) = \begin{cases} U(\overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, \widetilde{n}_1) - \widetilde{n}_1\widehat{e}_1, \widetilde{n}_1, U_1^D(\widehat{x}_1)), & \text{if } \widetilde{n}_1 \leq \widehat{n}_1, \\ U(\overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, \widetilde{n}_1) - \widehat{n}_1\widehat{e}_1 - (\widetilde{n}_1 - \widehat{n}_1)\,\widetilde{e}_1, \widetilde{n}_1, U_1^D(\widehat{x}_1)), & \text{if } \widetilde{n}_1 > \widehat{n}_1. \end{cases}.$$

Since $\widehat{a}$ is a Millian efficient allocation, it is straightforward to show that $G(\cdot)$ is strictly increasing and concave on $[0, \widehat{n}_1]$. Moreover, using first order conditions associated that every Millan efficient allocation must satisfy, we obtain that the left-hand side derivative of $G(\cdot)$ at $\widetilde{n}_1 = \widehat{n}_1$ is given by

$$\begin{aligned}\frac{d^- G(\widehat{n}_1)}{dn_1} &= b_0'(\widetilde{n}_1)\, D_1U(\widehat{x}, U_1(\widehat{x}_1)) + \left[D_2F_1(\widehat{k}_1^o, \widehat{n}_1) - \widehat{e}_1\right] D_2U(\widehat{x}, U_1^D(\widehat{x}_1)) + D_3U(\widehat{x}, U_1^D(\widehat{x}_1)) = \\ &= 0,\end{aligned}$$

while the right-hand side derivative is given by

$$\frac{d^+ G(\widehat{n}_1)}{dn_1} = \frac{d^- G(\widehat{n}_1)}{dn_1} + D_2U(\widehat{x}, U_1^D(\widehat{x}_1))\,(\widehat{e}_1 - \widetilde{e}_1) > 0.$$

Therefore,

$$G(\widetilde{n}_1) > G(\widehat{n}_1) = U_0(\widehat{x}),$$

for some $\widetilde{n}_1 > \widehat{n}_1$. Thus, if we select $\widetilde{n}_1 > \widehat{n}_1$, the allocation $\widetilde{\mathsf{a}}$ provides all agents already living with the allocation $\widehat{a}$ with at least the same utility than the utility they obtain with $\widehat{a}$ and provides some of these agents –the dynasty head– with strictly higher utility, which implies that $\widehat{a}$ is $\mathcal{A}-$inefficient and, therefore, a contradiction that establishes that $U_t^D(\widehat{x}) = V_t^D(\widehat{e}_t) = \mathcal{V}_t^D(\widehat{e}_t)$ for $t \geq 1$.

To complete the proof of Proposition 5, it only remains to be shown that $U_0(\widehat{x}) = V_0(\overline{e}_0) = \mathcal{V}_0(\overline{e}_0)$ must be satisfied for every symmetric allocation $\widehat{a}$. To show $U_0(\widehat{x}_0) = \mathcal{V}_0(\overline{e}_0)$ must be satisfied, suppose it is not. Taking into account that $\widehat{a}$ satisfies $U_1^D(\widehat{x}) = \mathcal{V}_1^D(\widehat{e}_1)$, this implies that $\widehat{e}_1 > e_1(\overline{e}_0)$, where $e_1(\overline{e}_0)$ is the solution to

$$\max\left\{W_0\left(\overline{e}_0, e_1, \mathcal{V}_1^D(e_1)\right) : e_1 \geq 0\right\}.$$

Choose now $\pi \in (0,1)$ and $\widetilde{e}_1 > 0$ be chosen in such a way that $e_1(\overline{e}_0) < \widetilde{e}_1 < \widehat{e}_1$ and

$$W_0(\overline{e}_0, \pi\widehat{e}_1 + (1-\pi)\,\widetilde{e}_1, \pi\mathcal{V}_1^D(\widehat{e}_1) + (1-\pi)\,\mathcal{V}_1^D(\widetilde{e}_1)) > W_0(\overline{e}_0, \widehat{e}_1, \mathcal{V}_1^D(\widehat{e}_1))$$

are both satisfied. Observe that the pair $(\pi, \widetilde{e}_1)$ must exist provided $e_1(\overline{e}_0) < \widetilde{e}_1 < \widehat{e}_1$ is satisfied.

Taking this into account, let $\widetilde{n}_1$ be arbitrary and consider the allocation $\widetilde{a}$ constructed from the pair $(\widetilde{n}_1, \widetilde{e}_1)$ as explained above and defined by $i) - iii)$. For such allocation we have

$$G(\widetilde{n}_1) = \begin{cases} U\left(\overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, \widetilde{n}_1) - \widetilde{n}_1\widehat{e}_1, \widetilde{n}_1, \mathcal{V}_1^D(\widehat{e}_1)\right), & \text{if } \widetilde{n}_1 \leq \widehat{n}_1, \\ U\left(\overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, \widetilde{n}_1) - \widehat{n}_1\widehat{e}_1 - (\widetilde{n}_1 - \widehat{n}_1)\,\widetilde{e}_1, \widetilde{n}_1, \frac{\widehat{n}_1}{\widetilde{n}_1}\mathcal{V}_1^D(\widehat{e}_1) + \left(1 - \frac{\widehat{n}_1}{\widetilde{n}_1}\right)\mathcal{V}_1^D(\widetilde{e}_1)\right), & \text{if } \widetilde{n}_1 > \widehat{n}_1. \end{cases}$$

Therefore, if $\widetilde{n}_1 > \widehat{n}_1$ is chosen in such a way that $\widetilde{n}_1\pi = \widehat{n}_1$ is satisfied, then we have

$$G(\widetilde{n}_1) = W_0(\overline{e}_0, \pi\widehat{e}_1 + (1-\pi)\,\widetilde{e}_1, \pi\mathcal{V}_1^D(\widehat{e}_1) + (1-\pi)\,\mathcal{V}_1^D(\widetilde{e}_1)) > W_0(\overline{e}_0, \widehat{e}_1, \mathcal{V}_1^D(\widehat{e}_1)) = G(\widehat{n}_1),$$

which implies that the allocation $\widetilde{a}$ $\mathcal{A}-$dominates the allocation $\widehat{a}$. Thus, the allocation $\widehat{a}$ is $\mathcal{A}-$inefficient, a contradiction that establishes that $U_0(\widehat{x}) = V_0(\overline{e}_0) = \mathcal{V}_0(\overline{e}_0)$ must be satisfied, which completes the proof of Proposition 5.□

Proof of Proposition 8. To prove Proposition 8, let $\mathcal{U}^N$ be a sequence of symmetric functions satisfying Properties 1 and 2, and let $\widehat{a}$ be a Millian efficient allocation such that the function $\mathbf{v}_0^D$ is concave on the set formed by all sequences $\{e_t : e_t > \widehat{e}_t\}_{t\geq 1}$. To show that $\widehat{a}$ is $\mathcal{P}$-efficient, let t be arbitrary and write $\mathbf{v}_t^D(\widehat{e}_t, \widehat{e}^{-t})$ as

$$\mathbf{v}_t^D(\widehat{e}_t, \widehat{e}^{-t}) = \max_{\mu_{t+1}:[\widehat{e}_{t+1},\infty]\to[0,1]} W_t^D\left(\widehat{e}_t, \int_{e\geq\widehat{e}_{t+1}} d\mu_{t+1}(e), \int_{e\geq\widehat{e}_{t+1}} \mathbf{v}_t^D(e, \widehat{e}^{-(t+1)})d\mu_{t+1}(e)\right),$$

which taking into account that $\mathbf{v}_t^D(\cdot, \widehat{e}^{-t})$ is concave on $[\widehat{e}_t, \infty]$ implies that the allocation that maximizes the utility of the dynasty head among all allocations satisfying $\mathsf{e}_t(i) \geq \widehat{e}_t$ for each t and each $i \in \mathcal{I}_t(\mathsf{n})$ is symmetric.

To show that this implies that $\widehat{a}$ must be $\mathcal{P}-$efficient, suppose it is not. That is, there exists an allocation a such that:

- All agents living in both a and $\widehat{a}$ obtain at least the same utility than the utility they obtain with $\widehat{a}$, that is
$$\mathcal{U}_t(\mathsf{x};i) \geq U_t(\widehat{x}) = v_t(\widehat{e}_t, \widehat{e}^{-t}) \text{ for every } i \in \mathcal{I}_t(n) \cap \mathcal{I}_t(\widehat{n}).$$

- All agents that are not born in $\widehat{a}$ but are born in a (that is, for which $\mathsf{n}_t(i^{t-1}, i) > \widehat{n}_t$) obtain, in the latter allocation, more utility than the utility attributed to the unborn in the allocation $\widehat{a}$. Since the utility attributed to the unborn satisfies Property 1 we have $\mathcal{U}_t^N(\widehat{\mathsf{x}}; i) = U_t(\widehat{x})$. Therefore,
$$\mathcal{U}_t(\mathsf{x};i) \geq U_t(\widehat{x}) = v_t(\widehat{e}_t, \widehat{e}^{-t}) \text{ for every } i \in \mathcal{I}_t(n) \text{ and } i \notin \mathcal{I}_t(\widehat{n}).$$

- All agents that were born in $\widehat{a}$ but are not born in a obtain more utility than the utility attributed to the unborn in a. By Property 2, we have
$$\mathcal{U}_t^N(\mathsf{x}; i^{t-1}, i) \geq \mathcal{U}_t(\mathsf{x};i) \geq U_t(\widehat{x}) = v_t(\widehat{e}_t, \widehat{e}^{-t}) \text{ if } \mathsf{n}_t(i^{t-1}) > 0.$$

Note that if the allocation $\widehat{a}$ is not $\mathcal{P}-$efficient, the allocation a^* that maximizes the utility of the dynasty head among all allocations a for which $\mathcal{U}_t(\mathsf{x};i) \geq U_t(\widehat{x})$ for each t and each $i \in \mathcal{I}_t(\mathsf{n})$ should differ from the allocation $\widehat{a}$. Proceeding as in Proposition 2, it is straightforward to show that such allocation a^* must maximize the utility of the dynasty head among all allocations satisfying $\mathsf{e}_t(i) \geq \widehat{e}_t$ for each t and each $i \in \mathcal{I}_t(\mathsf{n})$, which implies that the allocation a^* must be symmetric. But then the fact that a^* differs from $\widehat{a}$ implies that there exists a symmetric allocation that provides all generations of agents with at least the same utility than the utility they obtain with $\widehat{a}$, and with strictly higher utility to some of these generations, which contradicts the initial assumption imposing that $\widehat{a}$ is Millian efficient. This contradiction establishes that $\widehat{a}$ is Millian efficient, and completes the proof.□

Proof of Theorem 1. To prove theorem 1, let $\widehat{a}$ be a symmetric, decentralized equilibrium associated to a sequence of prices p. To show that $\widehat{a}$ is Millian efficient, we proceed by steps.

Step 1) First, observe that, for
$$p_{t+1} \quad = \quad (R_{t+1}, w_{t+1}) = \left(D_1 F_{t+1}(\widehat{k}_{t+1}, \widehat{n}_{t+1}), D_2 F_{t+1}(\widehat{k}_{t+1}, \widehat{n}_{t+1})\right),$$
first order conditions associated to the optimization problems in the definition of $W_{p,t}(\widehat{e}_t, \widehat{e}_{t+1})$ (or, respectively, $W_{p,t}^D(\widehat{e}_t, \widehat{e}_{t+1})$) coincide with those characterizing a solution to the optimization problem in the definition of $W_t(\widehat{e}_t, \widehat{e}_{t+1})$ (or, respectively, $W_t^D(\widehat{e}_t, \widehat{e}_{t+1})$). Therefore,
$$w_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) = w_t^D(\widehat{e}_t, \widehat{e}^{-t})$$
must be satisfied for all $t \geq 1$.

Step 2) In a Second step, we show that, since $\widehat{a}$ in a decentralized equilibrium, one must have, for $t \geq 1$,
$$w_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) = v_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) = \max\left\{w_{p,t}^D(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\}.$$
To prove Step 2) is satisfied, suppose it is not, that is, suppose $w_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) < v_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t})$ is satisfied for some t. Then proceed as in the proof of Proposition 2 to show that there exists an alternative path $\{\widetilde{e}_t\}_{t\geq 1}$ for which
$$w_{p,t}^D(\widetilde{e}_t, \widetilde{e}^{-t}) \geq w_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}),$$
and
$$W_{p,t}(\widetilde{e}_t, \widetilde{e}_{t+1}, w_{p,t+1}^D(\widetilde{e}_{t+1}, \widetilde{e}^{-(t+1)})) \geq W_{p,t}(\widehat{e}_t, \widehat{e}_{t+1}, w_{t+1}^D(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)}))$$
are satisfied for $t \geq 1$ -as strict inequalities for some period $\tau \geq 1$. But this contradicts the assumption imposing that $\{\widehat{e}_t\}_{t\geq 1}$ maximizes the utility of the dynasty head among all paths satisfying
$$W_{p,t}(e_t, e_{t+1}, w_{t+1}^D(e_t, e^{-(t+1)})) = \max\left\{W_t(e_t, e'_{t+1}, v^D(e'_{t+1}, e^{-(t+1)})) : e'_{t+1} \geq e_{t+1})\right\},$$

a contradiction that establishes Step 2).

Step 3) Third, we show that one must have, for $t \geq 1$,

$$w_t^D(\widehat{e}_t, \widehat{e}^{-t}) = v_t^D(\widehat{e}_t, \widehat{e}^{-t}) = \max\left\{w_t^D(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\}.$$

To prove Step 3) is satisfied, recall from 2 that $w_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) = v_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t})$ is satisfied for each $t \geq 1$. To show that $w_t^D(\widehat{e}_t, \widehat{e}^{-t}) = v_t^D(\widehat{e}_t, \widehat{e}^{-t})$ must be satisfied also for each $t \geq 1$, suppose not. That is, there exists $\{\widetilde{e}_t\}_{t\geq 1}$ and $t \geq 1$ for which $\widetilde{e}^{-t} \geq \widehat{e}^{-t}$ and $w_t^D(\widetilde{e}_t, \widetilde{e}^{-t}) \geq w_t^D(\widehat{e}_t, \widehat{e}^{-t})$ is satisfied. On the other hand, recall from the definitions that for each $t \geq 0$ and any (k_{t+1}, n_{t+1}) one must have

$$F_{t+1}(\widehat{k}_{t+1}, \widehat{n}_{t+1}) - R_{t+1}\widehat{k}_{t+1} - w_{t+1}\widehat{n}_{t+1} = 0 \geq F_{t+1}(k_{t+1}^o, n_{t+1}) - R_{t+1}k_{t+1}^o - w_{t+1}n_{t+1}.$$

Then,

$$F_{t+1}(k_{t+1}^o, n_{t+1}) - n_{t+1}e_{t+1} \leq R_{t+1}k_{t+1}^o + w_{t+1}n_{t+1}$$

and, therefore, the sequence $\left\{(\widehat{x}_\tau^p, \widehat{k}_\tau^p)\right\}_{\tau=t}^{\infty}$ that solve the sequence of optimization problems in the definition of $w_t^D(\widetilde{e}_t, \widetilde{e}^{-t})$ is feasible in the sequence of optimization problems in the definition of $w_{p,t}^D(\widetilde{e}_t, \widetilde{e}^{-t})$. This yields,

$$w_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) = w_t^D(\widehat{e}_t, \widehat{e}^{-t}) \geq w_{p,t}^D(\widetilde{e}_t, \widetilde{e}^{-t}) \geq w_t^D(\widetilde{e}_t, \widetilde{e}^{-t}),$$

for $t \geq 1$, a contradiction that establishes 3).

Step 4) In this step, we show that

$$U_t(\widehat{x}) = \max\left\{W_t(\widehat{e}_t, e_{t+1}, v^D(e_{t+1}, \widehat{e}^{-(t+1)})) : e_{t+1} \geq \widehat{e}_{t+1})\right\}$$

must be satisfied for $t \geq 0$. To prove Step 4), recall from the definitions and from Step 2) that

$$U_t(\widehat{x}) = \max\left\{W_{p,t}(\widehat{e}_t, e_{t+1}, v^{Dp}(e_{t+1}, \widehat{e}^{-(t+1)})) : e_{t+1} \geq \widehat{e}_{t+1})\right\}$$

must be satisfied for $t \geq 0$. By proceeding as in 3 we obtain, for every $t \geq 0$ and every $e_{t+1} \geq \widehat{e}_{t+1}$,

$$\begin{aligned} U_t(\widehat{x}) &= W_{p,t}(\widehat{e}_t, \widehat{e}_{t+1}, w_{p,t+1}^D(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, w_{t+1}^D(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)}) \\ &\geq W_{p,t}(\widehat{e}_t, e_{t+1}, v^D(e_{t+1}, \widehat{e}^{-(t+1)})) \\ &\geq W_t(\widehat{e}_t, e_{t+1}, v^D(e_{t+1}, \widehat{e}^{-(t+1)})), \end{aligned}$$

which establishes Step 4).

Step 5) In Steps 1)-4), we have shown that a symmetric allocation $\widehat{a}$ corresponding to a decentralized equilibrium satisfies the necessary conditions for Millian efficiency in Propositions 1 and 2 as well as the qualifying condition (13) in Proposition 3. By Proposition 3. $\widehat{a}$ must be statically $\mathcal{M}-$efficient. Moreover, Proposition 4 establishes that $\widehat{a}$ is dynamically $\mathcal{M}-$efficient if either (14) or (15) are satisfied, which completes the proof of Theorem 1.□

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